HALPHEN PENCILS ON WEIGHTED FANO THREEFOLD HYPERSURFACES

IVAN CHELTSOV AND JIHUN PARK

Abstract. We classify all pencils on a general weighted hypersurface of degree $\sum_{i=1}^{4} a_i$ in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ whose general members are surfaces of Kodaira dimension zero.

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Contents

Part 0. Introduction.

Throughout this article, all varieties are projective and defined over $\mathbb{C}$ and morphisms are proper, otherwise mentioned.

0.1. Introduction.

Let C be a smooth curve in $\mathbb{P}^2$ defined by an cubic homogeneous equation $f(x, y, z) = 0$. Suppose that we have nine distinct points $P_1, \cdots, P_9$ on C such that the divisor

$$\sum_{i=1}^{9} P_i - \mathcal{O}_{\mathbb{P}^2}(3)\Big|_C$$

is a torsion divisor of order $m \geq 1$ on the curve C. Then, there is a curve $Z \subset \mathbb{P}^2$ of degree $3m$ such that $\mathrm{mult}_{P_i}(Z) = m$ for each point P_i. Let $\mathcal{P}$ be the pencil given by the equation

$$\lambda f^m(x, y, z) + \mu g(x, y, z) = 0 \subset \mathrm{Proj}\Big(\mathbb{C}[x, y, z]\Big) \cong \mathbb{P}^2,$$

where $g(x, y, z) = 0$ is a homogeneous equation of the curve Z and $(\lambda : \mu) \in \mathbb{P}^1$. Then, a general curve of the pencil $\mathcal{P}$ is birational to an elliptic curve. The pencil $\mathcal{P}$ is called a plane Halphen pencil ([11]) and the construction of the pencil $\mathcal{P}$ can be generalized to the case when the curve C has ordinary double points and the points $P_1, \cdots, P_9$ are not necessarily distinct ([9]). In fact, every plane elliptic pencil is birational to a Halphen pencil. Namely, the following result is proved in [10] using the technique of plane Cremona transformations but its rigorous proof is due to [9].

Theorem 0.1.1. *Let $\mathcal{M}$ be a pencil on $\mathbb{P}^2$ whose general curve is birational to an elliptic curve. Then, there is a birational automorphism ρ of $\mathbb{P}^2$ such that $\rho(\mathcal{M})$ is a plane Halphen pencil.*

Proof. Every birational automorphism of $\mathbb{P}^2$ is a composition of projective automorphisms and Cremona involutions (Section 2.5 in [8]). Moreover, the arguments of Section 2.5 in [8] together with the two-dimensional analogue of Theorem 0.2.4 imply that there is birational automorphism $\rho \in \mathrm{Bir}(\mathbb{P}^2)$ such that the singularities of the log pair $(\mathbb{P}^2, \frac{3}{n}\mathcal{B})$ are canonical, where $\mathcal{B} = \rho(\mathcal{M})$ and n is the natural number such that $\mathcal{B} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(n)$.

The singularities of the log pair $(\mathbb{P}^2, \frac{3}{n}\mathcal{B})$ are not terminal by Theorem 0.2.4.[1] Therefore, there is a birational morphism $\pi : S \to \mathbb{P}^2$ such that the singularities of the log pair $(S, \frac{3}{n}\mathcal{B}_S)$ are terminal and

$$K_S + \frac{3}{n}\mathcal{B}_S \sim_{\mathbb{Q}} \pi^*\Big(K_{\mathbb{P}^2} + \frac{3}{n}\mathcal{B}\Big),$$

where $\mathcal{B}_S$ is the proper transform of the pencil $\mathcal{B}$ by the birational morphism π. Then, $-K_S$ is nef because the pencil $\mathcal{B}_S$ does not have any fixed curve. The divisor $-K_S$ is not big by Theorem 0.2.4, which implies that $K_S^2 = 0$. In particular, the complete linear system $|-nK_S|$ does not have fixed points. Therefore, the number n is divisible by 3 and

$$\rho(\mathcal{M}) = \pi\Big(\big|-\frac{n}{3}K_S\big|\Big),$$

which implies that $\rho(\mathcal{M})$ is a plane Halphen pencil. □

A problem similar to Theorem 0.1.1 can be considered for Fano varieties whose groups of birational automorphisms are well understood[2]. In particular, it would be very interesting to classify pencils of K3 surfaces on three-dimensional weighted Fano hypersurfaces.

Definition 0.1.2. A Halphen pencil is a one-dimensional linear system whose general element is birational to a smooth variety of Kodaira dimension zero.

[1] Theorem 0.2.4 can be generalized to each dimension ≥ 2.

[2] Elliptic pencils on some del Pezzo surfaces defined over an algebraically non-closed perfect field are birationally classified in [1] using the structure of their groups of birational automorphisms ([15]).

Let X be a general quasismooth hypersurface of degree $d=\sum_{i=1}^{4} a_i$ in $\mathbb{P}(1,a_1,a_2,a_3,a_4)$ that has terminal singularities, where $a_1\leq a_2\leq a_3\leq a_4$. Then,

$$-K_X\sim_{\mathbb{Q}}\mathcal{O}_{\mathbb{P}(1,\,a_1,\,a_2,\,a_3,\,a_4)}(1)\Big|_X,$$

which implies that X is a Fano threefold. The divisor class group $\mathrm{Cl}(X)$ is generated by the anticanonical divisor $-K_X$ and there are exactly 95 possibilities for the quadruple (a_1,a_2,a_3,a_4), which are found in [12]. We use the notation $\daleth$ for the entry numbers of these famous 95 families. They are ordered in the same way of [12], which is a standard way nowadays. We tabulate these families together with their properties in Part 6.

Birational geometry on such threefolds is extensively studied in [3], [4], [7], [20], and so forth. The article [7] describes the generators of the group $\mathrm{Bir}(X)$ of birational automorphisms of X. Also, the article [4] shows the relations among these generators. The former article proves the following result as well.

Theorem 0.1.3. *The threefold X cannot be rationally fibred by rational curves or surfaces.*

As for birational maps into elliptic fibrations, the hypersurface X in each family except the families of $\daleth=3$, 60, 75, 84, 87, and 93 is birational to an elliptic fibration ([4]). Furthermore, all birational transformations of the threefold X into elliptic fibrations are classified in [1], [2], [3], [4], and [20].

It is known that Halphen pencils on the threefold X always exist, to be precise, the threefold X can be always rationally fibred by K3 surfaces ([4]). In this article, we are to classify all Halphen pencils on the hypersurfaces in the 95 families as what is done for their elliptic fibrations in [3].

Let us explain five examples of pencils on the threefold X. They exhaust all the possible Halphen pencils on X. It follows from [7] that the pencils constructed below are $\mathrm{Bir}(X)$-invariant (Proposition 0.3.2). We will show, throughout this article, that they are indeed Halphen pencils.

Example I. Suppose that $a_2=1$. Then, every one-dimensional linear system in $|-K_X|$ is a Halphen pencil. It follows from the Adjunction that a general surface in $|-K_X|$ is birational to a smooth K3 surface. In particular, it belongs to Reid's 95 codimension 1 weighted K3 surfaces ([16]).

Therefore, in the cases in Example I, or equivalently $\daleth=1$, 2, 3, 4, 5, 6, 8, 10, 14, there are infinitely many Halphen pencils on the hypersurface X. Such cases will be studied in Part 5, where we will prove that every Halphen pencil is contained in $|-K_X|$.

Example II. Suppose that $a_1\neq a_2$. Then, the linear system $|-a_1K_X|$ is a pencil. If $a_1=1$, then the linear system $|-K_X|$ is a Halphen pencil and its general surface belongs to Reid's 95 codimension 1 weighted K3 surfaces as in Example I. In fact, it is a Halphen pencil if only if $a_1\neq a_2$ ([4]). We will see that it is a unique Halphen pencil except the cases with $a_2=1$ and the cases in three Examples below.

We will discuss the cases with a unique Halphen pencil in Parts 1 and 2.

Note that $a_1=a_2\neq 1$ exactly when $\daleth=18$, 22, and 28.

Example III. Suppose that $\daleth=18$, 22, or 28. In such cases, $a_1=a_2\neq 1$ and $a_3=a_1+1$. The threefold X has singular points $O_1,\cdots,O_r$ of type $\frac{1}{a_1}(1,1,a_1-1)$, where $r=\frac{3a_1+a_4+1}{a_1}$. There is a unique index $j\geq 3$ such that $a_2+a_3+a_4=ma_j$, where m is a natural number. In particular, the threefold X is given by an equation

$$\sum_{k=0}^{m} x_j^k f_k\big(x_0,x_1,x_2,x_3,x_4\big)=0\subset\mathrm{Proj}\Big(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]\Big),$$

where $\mathrm{wt}(x_0)=1$, $\mathrm{wt}(x_l)=a_l$, and f_k is a quasihomogeneous polynomial of degree $a_1+a_2+a_3+a_4-ka_j$ that is independent of the variable x_j. Let $\mathcal{P}_i$ be the pencil of surfaces in $|-a_1K_X|$ that pass through the point O_i and $\mathcal{P}$ be the pencil on the threefold X that is cut out by the

pencil $\lambda x_0^{a_1} + \mu f_m(x_0, x_1, x_2, x_3, x_4) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. It will be proved that $\mathcal{P}$ and $\mathcal{P}_i$ are Halphen pencils in $|-a_1K_X|$.

The cases in Example III are the only cases that have more than two finitely many but Halphen pencils. These cases will be discussed in Part 4.

Example IV. Suppose that $\gimel = 45, 48, 55, 57, 58, 66, 69, 74, 76, 79, 80, 81, 84, 86, 91, 93$, or 95. We then see $1 \neq a_1 \neq a_2$. Moreover, there is a unique index $j \neq 2$ such that $a_1 + a_3 + a_4 = ma_j$, where m is a natural number. Therefore, the threefold X is given by an equation

$$\sum_{k=0}^{m} x_j^k f_k\big(x_0, x_1, x_2, x_3, x_4\big) = 0 \subset \operatorname{Proj}\Big(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]\Big),$$

where $\mathrm{wt}(x_0) = 1$, $\mathrm{wt}(x_i) = a_i$, and f_k is a quasihomogeneous polynomial of degree $a_1 + a_2 + a_3 + a_4 - ka_j$ that is independent of the variable x_j. Let $\mathcal{P}$ be the pencil on the threefold X that is cut out by the pencil $\lambda x_0^{a_2} + \mu f_m(x_0, x_1, x_2, x_3, x_4) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. It will be shown that $\mathcal{P}$ is a Halphen pencil in $|-a_2K_X|$.

Example V. Suppose that $\gimel = 60$. Then, X is a general hypersurface of degree 24 in $\mathbb{P}(1, 4, 5, 6, 9)$. Hence, the threefold X is given by an equation

$$w^2 f_6\big(x, y, z, t\big) + wf_{15}\big(x, y, z, t\big) + f_{24}\big(x, y, z, t\big) = 0 \subset \operatorname{Proj}\Big(\mathbb{C}[x, y, z, t, w]\Big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 4$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$, and $f_k(x, y, z, t)$ is a general quasihomogeneous polynomial of degree k. Then, the linear system on the threefold X cut out by the pencil $\lambda x^6 + \mu f_6\big(x, y, z, t\big) = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$, is a Halphen pencil in $|-6K_X|$.

The cases in Examples IV and V have at least two Halphen pencils because they also satisfy the condition for Example II. Furthermore, we will see that these are the only Halphen pencils on the hypersurface X of each family. These cases are discussed in Part 3.

The main purpose of this article is to prove the following:[3].

Theorem 0.1.4. *Let X be a general hypersurface in the 95 families. Then, the pencils constructed in Examples I, II, III, IV, and V exhaust all possibilities for Halphen pencils on the hypersurface X.*

The following are immediate consequence of Theorem 0.1.4.

Corollary 0.1.5. *Let X be a general hypersurface in the 95 families with entry number $\gimel$.*

(1) *There are finitely many Halphen pencils on the threefold X if and only if $a_2 \neq 1$.*
(2) *There are at most two Halphen pencils on X in the case when $a_1 \neq a_2$.*
(3) *Every Halphen pencil on the threefold X is contained in $|-K_X|$ if $a_1 = 1$.*
(4) *The linear system $|-K_X|$ is the only Halphen pencil on X if $a_1 = 1$ and $a_2 \neq 1$.*
(5) *The linear system $|-a_1K_X|$ is the only Halphen pencil on the threefold X if and only if $\gimel \neq 1, 2, 3, 4, 5, 6, 8, 10, 14, 18, 22, 28, 45, 48, 55, 57, 58, 60, 66, 69, 74, 76, 79, 80, 81, 84, 86, 91, 93, 95$.*

Furthermore, Theorem 0.1.4 with Proposition 0.3.2 forces us to conclude

Corollary 0.1.6. *Let X be a general hypersurface in the 95 families. Then, every Halphen pencil on the threefold X is invariant under the action of $\mathrm{Bir}(X)$.*

The proof of Theorem 0.1.4 is based on Theorems 0.2.4, 0.2.9 and Lemmas 0.2.6, 0.2.7. We prove the theorem case by case in order of the number of Halphen pencils and the entry number $\gimel$.

In addition, we prove that general surfaces of the pencils constructed in Examples I, II, III, IV, V are birational to smooth K3 surfaces.

[3] Theorem 0.1.4 is proved in [19] and [20] for the cases $\gimel = 5, 34, 75, 88$, and 90.

Theorem 0.1.7. *Let X be a general hypersurface in the* 95 *families. Then, a general surface of every Halphen pencil on X is birational to a smooth K3 surface.*

It follows from Proposition 0.3.12 that general surfaces of the pencils constructed in Examples I and II are birational to smooth K3 surfaces. General surfaces of the other pencils will be discussed in Part 3. The proof of Theorem 0.1.7 is based on Corollaries 0.2.11 and 0.2.12.

Theorems 0.1.4 and 0.1.7 tell us how a general hypersurface in the 95 families can be rationally fibred by smooth surfaces of Kodaira dimension zero.

Corollary 0.1.8. *Let X be a general hypersurface in the* 95 *families and let $\pi : Y \to Z$ be a morphism whose general fiber is birational to a smooth surface of Kodaira dimension zero. If there is a birational map $\alpha : X \dashrightarrow Y$, then there is an isomorphism $\phi : \mathbb{P}^1 \to Z$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, dashed, "\alpha"] \arrow[d, dashed, "\psi"'] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\phi"'] & Z, \end{tikzcd}$$

where the rational map $\psi : X \dashrightarrow \mathbb{P}^1$ is induced by one of the pencils in Examples I, II, III, IV, and V. In particular, a general fiber of the morphism π is birational to a smooth K3 surface.

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0.2. Preliminaries.

Let X be a threefold with $\mathbb{Q}$-factorial singularities and $\mathcal{M}$ be a linear system on the threefold X without fixed components. We consider the log pair $(X, \mu\mathcal{M})$ for some nonnegative rational number μ.

Let $\alpha : Y \to X$ be a proper birational morphism such that Y is smooth and the proper transform $\mathcal{M}_Y$ of the linear system $\mathcal{M}$ by the birational morphism α is base-point-free. Then, the rational equivalence

$$K_Y + \mu\mathcal{M}_Y \sim_{\mathbb{Q}} \alpha^*\Big(K_X + \mu\mathcal{M}\Big) + \sum_{i=1}^{k} a_i E_i$$

holds, where E_i is an exceptional divisor of the birational morphism α and a_i is a rational number.

Definition 0.2.1. The singularities of the log pair $(X, \mu\mathcal{M})$ are terminal (canonical, log-terminal, respectively) if each rational number a_i is positive (nonnegative, greater than -1, respectively). In case, we also say that the log pair $(X, \mu\mathcal{M})$ is terminal (canonical, log-terminal, respectively).

It is convenient to specify where the log pair $(X, \mu\mathcal{M})$ is not terminal.

Definition 0.2.2. A proper irreducible subvariety $Z \subset X$ is called a center of canonical singularities of the log pair $(X, \mu\mathcal{M})$ if there is an exceptional divisor E_i such that $\alpha(E_i) = Z$ and $a_i \leq 0$. The set of all proper irreducible subvarieties of X that are centers of canonical singularities of the log pair $(X, \mu\mathcal{M})$ is denoted by $\mathbb{CS}(X, \mu\mathcal{M})$.

A curve not contained in the singular locus of the threefold X is a center of canonical singularities of the log pair $(X, \mu\mathcal{M})$ if and only if the multiplicity of a general surface of $\mathcal{M}$ along the curve is not smaller than $\frac{1}{\mu}$. Furthermore, we obtain

Lemma 0.2.3. *Let C be a curve on the threefold X that is not contained in the singular locus of X. Suppose that the curve C is a center of canonical singularities of the log pair $(X, \mu\mathcal{M})$ and the linear system $|-mK_X|$ is base-point-free for some natural number $m > 0$. If $-K_X \sim_{\mathbb{Q}} \mu\mathcal{M}$, then $-K_X \cdot C \leq -K_X^3$.*

Proof. Let M_1 and M_2 be general surfaces in $\mathcal{M}$. Then,

$$\mathrm{mult}_C(M_1 \cdot M_2) \geq \mathrm{mult}_C(M_1)\mathrm{mult}_C(M_2) \geq \frac{1}{\mu^2}.$$

Let H be a general surface in $|-mK_X|$. Then,

$$\frac{m}{\mu^2}(-K_X^3) = H \cdot M_1 \cdot M_2 \geq (-mK_X \cdot C)\mathrm{mult}_C(M_1 \cdot M_2) \geq \frac{m}{\mu^2}(-K_X \cdot C),$$

which implies $-K_X \cdot C \leq -K_X^3$. □

The following result is a generalization of so-called Noether–Fano inequality ([8]).

Theorem 0.2.4. *Suppose that the linear system $\mathcal{M}$ is a pencil whose general surface is birational to a smooth surface of Kodaira dimension zero, the linear system $|-mK_X|$ is base-point-free for some natural m, and $\mathcal{M} \sim_{\mathbb{Q}} -\mu K_X$. If the linear system $|-mK_X|$ induces either a birational morphism or an elliptic fibration, then the log pair $(X, \mu\mathcal{M})$ is not terminal.*

Proof. Let M be a general surface in $\mathcal{M}$. Suppose that the log pair $(X, \mu M)$ is terminal. Then, for some positive rational number $\epsilon > \mu$, the log pair $(X, \epsilon M)$ is also terminal and the divisor $K_X + \epsilon M$ is nef. We have a resolution of indeterminacy of the rational map $\rho : X \dashrightarrow \mathbb{P}^1$ induced by the pencil $\mathcal{M}$ as follows:

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^1, \end{tikzcd}$$

where Y is smooth and β is a morphism. We consider the linear equivalence

$$K_Y + \epsilon M_Y \sim_{\mathbb{Q}} \alpha^*\Big(K_X + \epsilon M\Big) + \sum_{i=1}^{k} c_i E_i,$$

where M_Y is the proper transform of the surface M and c_i is a rational number. Then, each c_i is positive. Also, we may assume that the proper transform $\mathcal{M}_Y$ of the pencil $\mathcal{M}$ by the birational morphism α is base-point-free. In particular, the surface M_Y is smooth.

Let l be a sufficiently big and divisible natural number. Then, the negativity property of the exceptional locus of a birational morphism (Section 1.1 in [21]) implies that the linear system $|l(K_Y + \epsilon M_Y)|$ gives a dominant rational map $\xi : Y \dashrightarrow V$ with $\dim(V) \geq 2$. One the other hand, since the proper transform $\mathcal{M}_Y$ is a base-point-free pencil, the Adjunction formula implies

$$l\Big(K_Y + \epsilon M_Y\Big)\Big|_{M_Y} \sim lK_{M_Y}.$$

However, the surface M_Y has Kodaira dimension zero, which implies that $\dim(V) \leq 1$. It is a contradiction. □

Theorem-Definition 0.2.5. *Let $(P \in U)$ be the germ of a threefold terminal quotient singularity P of type $\frac{1}{r}(1, a, r-a)$, where $r \geq 2$, $r > a$, and a is coprime to r. Suppose that*

$$f : (E \subset W) \to (Z \subset U)$$

is a proper birational morphism such that

- *the threefold W has at worst terminal singularities;*
- *the exceptional set E of f is an irreducible divisor of W;*
- *the divisor $-K_W$ is f-ample;*
- *the point P is contained in the subvariety Z.*

Then, it is the weighted blow up at the point P with weights $(1, a, r-a)$. In particular, $Z = P$. We call such a birational morphism the Kawamata blow up at the point P with weights $(1, a, r-a)$ or simply the Kawamata blow up at the point P.

Proof. See [13]. □

Let $\pi : Y \to X$ be the Kawamata blow up at a quotient singular point P of type $\frac{1}{r}(1, a, r-a)$, where $r \geq 2$, $r > a$, and a is coprime to r. One can easily check that the exceptional divisor E of the birational morphism π is isomorphic to $\mathbb{P}(1, a, r-a)$. Furthermore, we see

$$\begin{cases} K_Y = \pi^*(K_X) + \dfrac{1}{r}E, \\ K_Y^3 = K_X^3 + \dfrac{1}{ra(r-a)}, \\ E^3 = \dfrac{r^2}{a(r-a)}. \end{cases}$$

Otherwise mentioning, from this point throughout this section, we always assume that the linear system $\mathcal{M}$ is a pencil with $\mathcal{M} \sim_{\mathbb{Q}} -\mu K_X$. In addition, we always assume that a general surface of the pencil $\mathcal{M}$ is irreducible.

For our purpose Theorem-Definition 0.2.5 can be modified as follows.

Lemma 0.2.6. *Let P be a singular point of a threefold X that is a quotient singularity of type $\frac{1}{r}(1, a, r-a)$, $r \geq 2$, $r > a$, and a is coprime to r. Suppose that the log pair $(X, \mu\mathcal{M})$ is canonical but the set $\mathbb{CS}(X, \mu\mathcal{M})$ contains either the point P or a curve passing through the point P. Let $\pi : Y \to X$ be the Kawamata blow up at the point P and let $\mathcal{M}_Y$ be the proper transform of $\mathcal{M}$ by the birational morphism π. Then,*

$$\mu\mathcal{M}_Y \sim_{\mathbb{Q}} \pi^*\Big(-K_X\Big) - \frac{1}{r}E \sim_{\mathbb{Q}} -K_Y,$$

where E is the exceptional divisor of π.

Proof. We consider only the case $r = 2$. Then, $E \cong \mathbb{P}^2$ and $E|_E \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(-2)$. We have

$$\mathcal{M}_Y \sim_{\mathbb{Q}} \pi^*\Big(-\frac{1}{\mu}K_X\Big) - mE,$$

where m is a positive rational number. In particular, we have $\mathcal{M}_Y|_E \equiv -mE|_E$.

Suppose that $m < \frac{1}{2\mu}$. Let Q be a point of E. Intersecting a general divisor of the pencil $\mathcal{M}_Y$ with a general line on E that passes through the point Q, we obtain the inequality

$$\mathrm{mult}_Q(\mathcal{M}_Y) \leq 2m < \frac{1}{\mu}.$$

Suppose that the set $\mathbb{CS}(X, \mu\mathcal{M})$ contains an irreducible curve Z that passes through the point P. Then,

$$\frac{1}{\mu} > \mathrm{mult}_O\big(\mathcal{M}_Y\big) \geq \mathrm{mult}_{Z_Y}\big(\mathcal{M}_Y\big) \geq \mathrm{mult}_Z\big(\mathcal{M}\big) \geq \frac{1}{\mu},$$

where Z_Y is the proper transform of the curve Z and O is a point of the intersection of the curve Z_Y and the exceptional divisor E. Therefore, the singularities of the log pair $(X, \mu\mathcal{M})$ are terminal in a punctured neighborhood of the point P. The equivalence

$$K_Y + \mu\mathcal{M}_Y \sim_{\mathbb{Q}} \pi^*\Big(K_X + \mu\mathcal{M}\Big) + \Big(\frac{1}{2} - \mu m\Big)E,$$

shows that the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$ contains a proper subvariety $\Delta \subsetneq E$. It implies that the inequality $\mathrm{mult}_\Delta(\mathcal{M}_Y) \geq \frac{1}{\mu}$ holds, which is a contradiction. □

Lemma 0.2.6 can be generalized in the following way ([3]).

Lemma 0.2.7. *Under the assumptions and notations of Lemma 0.2.6, suppose that we have a proper subvariety* $Z \subset E \cong \mathbb{P}(1, a, r-a)$ *that belongs to* $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$. *Then, the following hold:*

1. *The subvariety* Z *is not a smooth point of the surface* E*;*
2. *If the subvariety* Z *is a curve, then it belongs to the linear system* $|\mathcal{O}_{\mathbb{P}(1,\,a,\,r-a)}(1)|$ *defined on the surface* E *and all singular points of the surface* E *are contained in the set* $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$.

Proof. We consider only the case when $r = 5$ and $a = 2$ because the proofs for the other cases are very similar. Thus, we have $E \cong \mathbb{P}(1, 2, 3)$. Let Q_1 and Q_2 be the singular points of the surfaces E and L be the unique curve in $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$ on the surface E. Then, L contains the singular points Q_1 and Q_2 but the equivalence $\mu\mathcal{M}_Y|_E \equiv L$ holds by Lemma 0.2.6. Also, it follows from Lemma 0.2.6 that the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$ contains both the points Q_1 and Q_2 if the curve L is contained in the set $\mathbb{CS}(Y, \mu\mathcal{M}_Y)$.

Suppose that the subvariety Z is different from L, Q_1, and Q_2. Let us show that this assumption gives us a contradiction.

Suppose that Z is a point. Then, Z is a smooth point of the threefold Y, which implies the inequality $\mathrm{mult}_Z(\mathcal{M}_Y) > \frac{1}{\mu}$. Let C be a general curve in $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(6)|$ on the surface E that passes through the point Z. Then, C is not contained in the base locus of $\mathcal{M}_Y$, which implies the following contradictory inequality:

$$\frac{1}{\mu} = C \cdot \mathcal{M}_Y \geq \mathrm{mult}_Z\big(\mathcal{M}_Y\big) > \frac{1}{\mu}.$$

Therefore, the subvariety Z must be a curve. Then, $\mathrm{mult}_Z(\mathcal{M}_Y) \geq \frac{1}{\mu}$. Let C be a general curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(6)|$ on the surface E. Then, the curve C is not contained in the base locus of the pencil $\mathcal{M}_Y$. Therefore, we have

$$\frac{1}{\mu} = C \cdot \mathcal{M}_Y \geq \mathrm{mult}_Z\big(\mathcal{M}_Y\big) C \cdot Z \geq \frac{1}{\mu} C \cdot Z,$$

which implies that $C \cdot Z = 1$ on the surface E. The equality $C \cdot Z = 1$ implies that the curve Z is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$ on the surface E, which is impossible due to our assumption. □

Lemma 0.2.8. *Let* S *be a normal surface and* Δ *be an effective divisor on* S *such that*

$$\Delta \equiv \sum_{i=1}^{r} a_i C_i,$$

where $C_1, \cdots, C_r$ *are irreducible curves on* S *and* a_i *is a rational number. If the intersection form of the curves* $C_1, \cdots, C_r$ *on the surface* S *is negative-definite, then* $\Delta = \sum_{i=1}^{r} a_i C_i$.

Proof. Let $\Delta = \sum_{i=1}^{k} c_i B_i$, where B_i is an irreducible curve on the surface S and c_i is a nonnegative rational number. Suppose that

$$\sum_{i=1}^{k} c_i B_i \neq \sum_{i=1}^{r} a_i C_i.$$

Then, we may assume that each curve B_i coincides with none of the curves $C_1, \cdots, C_r$. We have

$$\begin{aligned} 0 &\geq \Big(\sum_{a_i>0} a_i C_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) \\ &= \Big(\sum_{i=1}^{k} c_i B_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) - \Big(\sum_{a_i\leq 0} a_i C_i\Big) \cdot \Big(\sum_{a_i>0} a_i C_i\Big) \geq 0, \end{aligned}$$

which immediately implies

$$\sum_{a_i>0} a_i C_i = 0.$$

Therefore, we obtain the numerical equivalence

$$\sum_{i=1}^{k} c_i B_i \equiv \sum_{a_i \leq 0} a_i C_i.$$

It then follows that $c_i = 0$ and $a_i = 0$ for every i. □

The following result is a generalization of Lemma A.20 in [3].

Theorem 0.2.9. *Let $\mathcal{B}$ be a linear system on a threefold X such that general surface of the linear system $\mathcal{B}$ is irreducible. Then, the linear system $\mathcal{B}$ coincides with the pencil $\mathcal{M}$ if one of the following holds:*

(0) *There is a Zariski closed proper subset $\Sigma \subset X$ such that*

$$\mathrm{Supp}\big(M\big) \cap \mathrm{Supp}\big(B\big) \subseteq \Sigma,$$

where M and B are general divisors of the pencil $\mathcal{M}$ and the linear system $\mathcal{B}$, respectively. Note that the general divisors M and B are chosen independently of the proper subset Σ.

For the below, let B and M be general surfaces of the linear system $\mathcal{B}$ and the pencil $\mathcal{M}$, respectively.

(1) *There is a nef and big divisor D on the threefold X such that $D \cdot M \cdot B = 0$.*
(2) *The base locus of $\mathcal{B}$ consists of an irreducible curve C such that $M \cdot B \equiv \lambda C$ and $B \cdot C < 0$ for some positive rational number λ.*
(3) *The base locus of $\mathcal{M}$ consists of an irreducible curve C such that $M \cdot B \equiv \lambda C$ and $M \cdot C < 0$ for some positive rational number λ.*
(4) *The equivalence $M \equiv \lambda B$ holds for some positive rational number λ and the base locus of $\mathcal{B}$ consists of an irreducible curve C such that $B \cdot C < 0$.*
(5) *The equivalence $M \equiv \lambda B$ holds for some positive rational number λ and the base locus of $\mathcal{M}$ consists of an irreducible curve C such that $M \cdot C < 0$.*
(6) *The surface B is normal, the equivalence $M \equiv \lambda B$ holds for some positive rational number λ, and the base locus of $\mathcal{B}$ consists of irreducible curves $C_1, \cdots, C_r$ whose intersection form on the surface B is negative-definite.*
(7) *The surface M is normal, the equivalence $M \equiv \lambda B$ holds for some positive rational number λ, and the base locus of $\mathcal{M}$ consists of irreducible curves $C_1, \cdots, C_r$ whose intersection form on the surface M is negative-definite.*

Proof. (0). Let $\rho : X \dashrightarrow \mathbb{P}^1$ be the rational map induced by the pencil $\mathcal{M}$ and $\xi : X \dashrightarrow \mathbb{P}^r$ be the rational map induced by the linear system $\mathcal{B}$. We then consider a simultaneous resolution of both the rational maps as follows:

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[d, "\pi"] \arrow[dr, "\beta"] & \\ \mathbb{P}^r & X \arrow[l, dashed, "\xi"] \arrow[r, dashed, "\rho"'] & \mathbb{P}^1, \end{tikzcd}$$

where W is a smooth variety, π is a birational morphism, α and β are morphisms.

Let Λ be a Zariski closed subset of the variety W such that the morphism

$$\pi\Big|_{W \setminus \Lambda} : W \setminus \Lambda \longrightarrow X \setminus \pi\big(\Lambda\big)$$

is an isomorphism and Δ be the union of the set Λ and the closure of the proper transform of the set $\Sigma \setminus \pi(\Lambda)$ on the variety W. Then, the set Δ is a proper subvariety of W.

Suppose that the pencil $\mathcal{M}$ is different from the linear system $\mathcal{B}$. Let B_W be the pull-back of a general hyperplane of $\mathbb{P}^r$ by the morphism α and let M_W be a general fiber of the morphism β. Then, the intersection $M_W \cap B_W$ is not empty and the support $\mathrm{Supp}(M_W \cap B_W)$ is not contained in Δ. Hence, we have

$$\mathrm{Supp}\Big(\pi\big(M_W\big)\Big) \cap \mathrm{Supp}\Big(\pi\big(B_W\big)\Big) \not\subseteq \Sigma,$$

where $\pi(M_W)$ and $\pi(B_W)$ are general divisors in the linear systems $\mathcal{M}$ and $\mathcal{B}$, respectively.

(1). For some positive number m, there is an ample divisor A and an effective divisor E such that $mD \sim A + E$ since the divisor D is nef and big. Then, the inequality $E \cdot M \cdot B < 0$ follows from $mD \cdot M \cdot B = 0$ and $A \cdot M \cdot B > 0$. Therefore, the support of the cycle $M \cdot B$ must be contained in the support of the effective divisor E, and hence the statement (0) completes the proof.

(2). Let H be an ample divisor on X. Then, there is a positive rational number ϵ such that $(B + \epsilon H) \cdot C = 0$. Since $M \cdot B \equiv \lambda C$, we obtain $(B + \epsilon H) \cdot M \cdot B = 0$. Because the divisor $B + \epsilon H$ is nef and big, (1) implies $\mathcal{M} = \mathcal{B}$.

(3). The proof is the same as (2).

(4). It is an immediate consequence of (2).

(5). It is an immediate consequence of (3).

(6). Since $M \equiv \lambda B$, the restricted divisor $M|_B$ of the effective divisor M to the surface B is numerically equivalent to the divisor $\sum_{i=1}^r a_i C_i$ on the normal surface B, where a_i is a positive rational number. It then follows from Lemma 0.2.8 that

$$M\Big|_B = \sum_{i=1}^r a_i C_i.$$

It implies that

$$\mathrm{Supp}\big(M\big) \cap \mathrm{Supp}\big(B\big) \subset \mathrm{Supp}\Big(\sum_{i=1}^r a_i C_i\Big).$$

Then, the statement (0) completes the proof.

(7). The proof is the same as (6). □

Theorem 0.2.10. *Suppose that a log pair $(X, \mu\mathcal{M})$ is canonical with $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} 0$. In addition, we suppose that one of the following hold:*

1. *the base locus of the pencil $\mathcal{M}$ consists of irreducible curves $C_1, \cdots, C_r$ and there is a nef and big divisor D on X such that $D \cdot C_i = 0$ for each i;*
2. *the base locus of $\mathcal{M}$ consists of an irreducible curve C such that $\mathcal{M} \cdot C < 0$;*
3. *a general surface of the pencil $\mathcal{M}$ is normal, the base locus of $\mathcal{M}$ consists of irreducible curves $C_1, \cdots, C_r$ whose intersection form is negative-definite on a general surface in the pencil $\mathcal{M}$.*

Then, the linear system $\mathcal{M}$ is a Halphen pencil and there is a composition of antiflips $\xi : X \dashrightarrow X'$ along the curves $C_1, \cdots, C_r$ (or C) such that the proper transform $\mathcal{M}_{X'}$ of the pencil $\mathcal{M}$ by ξ is base-point-free.

Proof. The log pair $(X, \lambda\mathcal{M})$ is log-terminal for some rational number $\lambda > \mu$. Hence, it follows from [21] that there is a birational map $\xi : X \dashrightarrow X'$ such that ξ is an isomorphism in codimension one, the log pair $(X', \lambda\mathcal{M}_{X'})$ is log-terminal, and the divisor $K_{X'} + \lambda\mathcal{M}_{X'}$ is nef.

Let H be a general surface in the pencil $\mathcal{M}_{X'}$. Since

$$H \equiv \frac{1}{\lambda - \mu}\Big(K_{X'} + \lambda\mathcal{M}_{X'} - (K_X' + \mu\mathcal{M}_{X'})\Big),$$

the divisor H is nef. Hence, it follows from the log abundance theorem ([14]) that the linear system $|mH|$ is base-point-free for some $m \gg 0$.

Let $\mathcal{B}$ be the proper transform of the linear system $|mH|$ on X. Also, let B and M be general surfaces of the linear system $\mathcal{B}$ and the pencil $\mathcal{M}$, respectively. Then, $B \equiv mM$ and one of the

conditions in Theorem 0.2.9 is satisfied. Hence, we have $\mathcal{M} = \mathcal{B}$, which implies that $m = 1$ and $\mathcal{M}_{X'} = |H|$ is base-point-free and induces a morphism $\pi' : X' \to \mathbb{P}^1$. Thus, every member of the pencil $\mathcal{M}_{X'}$ is contracted to a point by the morphism π'.

The log pair $(X', \mu\mathcal{M}_{X'})$ is canonical because the map ξ is a log flop with respect to the log pair $(X, \mu\mathcal{M})$. In particular, the singularities of X' are canonical. Hence, the surface H has at most Du Val singularities because the pencil $\mathcal{M}_{X'}$ is base-point-free. Moreover, the equivalence $K_{X'} + \mu H \sim_{\mathbb{Q}} 0$ and the Adjunction formula imply that $K_H \sim 0$. Consequently, the linear system $\mathcal{M}$ is a Halphen pencil. □

Corollary 0.2.11. *Under the assumption and notations of Theorem 0.2.10, in addition, suppose that a general surface of the pencil $\mathcal{M}$ is linearly equivalent to $-nK_X$ for some natural number n. Then, a general element of $\mathcal{M}$ is birational either to a smooth K3 surface or to an abelian surface.*

Proof. It immediately follows from the proof of Theorem 0.2.10 and the classification of smooth surfaces of Kodaira dimension zero. □

Corollary 0.2.12. *Under the assumption and notations of Corollary 0.2.11, suppose that a general surface of the pencil $\mathcal{M}$ has a rational curve not contained in the base locus of the pencil $\mathcal{M}$. Then, a general element of $\mathcal{M}$ is birational to a smooth K3 surface.*

Proof. In the proof of Theorem 0.2.10, suppose that the surface M has a rational curve L not contained in the base locus of the pencil $\mathcal{M}$. Then, the surface H contains a rational curve because the birational map ξ makes no change along the curve L. On the other hand, the surface H is birational either to a smooth K3 surface or to an abelian surface. However, an abelian surface cannot contain a rational curve. □

0.3. General results.

Let X be a general hypersurface of the 95 families with entry number $\beth$ in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$. In addition, let $\mathcal{M}$ be a Halphen pencil on the threefold X. Then, the log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal by Theorem 0.2.4, where n is the natural number such that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$. The following result is due to [7].

Theorem 0.3.1. *There is a birational automorphism $\tau \in \mathrm{Bir}(X)$ such that the log pair $(X, \frac{1}{m}\tau(\mathcal{M}))$ is canonical, where m is the natural number such that $\tau(\mathcal{M}) \sim_{\mathbb{Q}} -mK_X$.*

To classify Halphen pencils on X up to the action of $\mathrm{Bir}(X)$, we may assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical. However, it is not terminal by Theorem 0.2.4.

Proposition 0.3.2. *The pencils constructed in Examples I, II, III, IV, and V are invariant under the action of the group $\mathrm{Bir}(X)$ of birational automorphisms of X.*

Proof. Suppose that the hypersurface X is defined by the equation

$$f_d(x, y, z, t, w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, and f_d is a general quasihomogeneous polynomial of degree $d = \sum a_i$.

Since the hypersurface X is general, it is not hard to see that the group $\mathrm{Aut}(X)$ of automorphisms of X is either trivial or isomorphic to the group of order 2. The latter case happens when $2a_4 = d$. In such a case, the hypersurface X can be defined by an equation of the form

$$w^2 = g_d(x, y, z, t),$$

where g_d is a general quasihomogeneous polynomial of degree d in variables x, y, z, and t. The group $\mathrm{Aut}(X)$ is generated by the involution $[x : y : z : t : w] \mapsto [x : y : z : t : -w]$. Therefore, in both the cases, we can see that the pencils constructed in Examples I, II, III, IV, and V are invariant under the action of the group $\mathrm{Aut}(X)$ of automorphisms of X.

Suppose that the hypersurface X is not superrigid, i.e., it has a birational automorphism that is not biregular. Then, it is either a quadratic involution or an elliptic involution that are described in [7]. A quadratic involution has no effect on things defined with the variables x, y, z, and t (see Theorem 4.9 in [7]). On the other hand, an elliptic involution has no effect on things defined with the variables x, y, and z (see Theorem 4.13 in [7]). The pencils constructed in Examples I, II, III, and IV are defined by the variables x, y, and z. Therefore, such pencils are invariant under the action of the group $\mathrm{Bir}(X)$ of birational automorphisms of X. Meanwhile, the pencil constructed in Example V is contained in $|-a_3K_X|$. However, in the case $\gimel = 60$, the hypersurface X does not have an elliptic involution (see The Big Table in [7]), and hence the pencil is also $\mathrm{Bir}(X)$-invariant. □

Every Halphen pencil on the threefold X is, as shown throughout the present article, birational to a pencil in Examples I, II, III, IV, and V that is $\mathrm{Bir}(X)$-invariant. It implies that every Halphen pencil on X is $\mathrm{Bir}(X)$-invariant.

Lemma 0.3.3. *Suppose that $\gimel \geq 3$. Then, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains no smooth point of X.*

Proof. It follows from the proof of Theorem 5.3.1 in [7]. □

Corollary 0.3.4. *Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve C. Then, $-K_X\cdot C \leq -K_X^3$.*

Proof. It is an immediate consequence of Lemma 0.2.3. □

Corollary 0.3.5. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point of X whenever $\gimel \geq 7$.*

Proof. If $\gimel \geq 7$, then $-K_X^3 < 1$. Therefore, each curve C with $-K_X\cdot C \leq -K_X^3$ passes through a singular point of X. Then, the result follows from Lemmas 0.2.6, 0.3.3, and Corollary 0.3.4. □

In fact, we have a stronger result as follows:

Theorem 0.3.6. *Suppose that $\gimel \geq 3$ and the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve C. Then,*

$$\mathrm{Supp}(C) \subset \mathrm{Supp}\Big(S_1 \cdot S_2\Big),$$

where S_1 and S_2 are distinct surfaces of the linear system $|-K_X|$.

Proof. See Section 3.1 in [20]. □

Corollary 0.3.7. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains no curves whenever $a_1 \neq 1$.*

Corollary 0.3.8. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve and $a_2 \neq 1$, then $\mathcal{M} = |-K_X|$.*

Proof. It follows from Theorem 0.3.6 and Theorem 0.2.9. □

Suppose that $\gimel \geq 7$. Then, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point P of type $\frac{1}{r}(1, r-a, a)$, where $r \geq 2$, $r > a$, and a is coprime to r. Let $\pi : Y \to X$ be the Kawamata blow up at the singular point P and E be its exceptional divisor. In addition, let $\mathcal{M}_Y$ be the proper transform of the pencil $\mathcal{M}$ by the birational morphism π. Then, $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ by Lemma 0.2.6. The cone $\overline{\mathbb{NE}}(Y)$ of the threefold Y contains two extremal rays R_1 and R_2 such that π is a contraction of the extremal ray R_1. Moreover, the following result holds.

Proposition 0.3.9. *Suppose that $-K_Y^3 \leq 0$ and $\gimel \neq 82$. Then, the threefold Y contains irreducible surfaces $S \sim_{\mathbb{Q}} -K_Y$ and $T \sim_{\mathbb{Q}} -bK_Y + cE$ whose scheme-theoretic intersection is an irreducible reduced curve that generates R_2, where $b > 0$ and $c \geq 0$ are integer numbers.*

Proof. See Lemma 5.4.3 in [7]. □

The values of b and c for a given singular point in Proposition 0.3.9 appear in The Table of Part 6.

Lemma 0.3.10. *Under the assumptions and notations of Proposition 0.3.9, suppose that the inequality $-K_Y^3 < 0$ holds. Then, the number c is zero.*

Proof. Let M_1 and M_2 be general surfaces of the pencil $\mathcal{M}_Y$. Then, $M_1 \cdot M_2 \equiv n^2 K_Y^2$, which implies that $M_1 \cdot M_2 \notin \overline{\mathbb{NE}}(Y)$ in the case $c > 0$ by Proposition 0.3.9. ☐

Lemma 0.3.11. *Under the assumptions and notations of Proposition 0.3.9, suppose that the inequality $-K_Y^3 < 0$ holds. Then, the pencil $\mathcal{M}_Y$ is generated by the divisors bS and T.*

Proof. Let M_1 and M_2 be general surfaces in $\mathcal{M}_Y$. Then, $M_1 \cdot M_2 \in \overline{\mathbb{NE}}(Y)$ and $M_1 \cdot M_2 \equiv n^2 K_Y^2$, which implies that $M_1 \cdot M_2 \in \mathbb{R}^+ R_2$ because $c = 0$ by Lemma 0.3.10. Moreover, we have

$$\mathrm{Supp}(\Gamma) = \mathrm{Supp}\Big(M_1 \cdot M_2\Big),$$

because $T \cdot R_2 < 0$ and $S \cdot R_2 < 0$. Therefore, the pencil $\mathcal{M}_Y$ coincides with the pencil generated by the divisors bS and T by Theorem 0.2.9, which completes the proof. ☐

When $a_1 = 1$, a general surface of a pencil contained in the linear system $|-K_X|$ is birational to a K3 surface. In particular, it is one of Reid's 95 codimension 1 weighted K3 surfaces. Furthermore, we have the following:

Proposition 0.3.12. *In every case, a general surface in a pencil contained in $|-a_1K_X|$ is birational to a K3 surface.*

Proof. See [4][4] ☐

Therefore, general surfaces in the pencils of Examples I and II are birational to K3 surfaces.

0.4. Notations.

Let us describe the notations we will use. Otherwise mentioned, these notations are fixed from Part 1 to Part 6.

- In the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$. For weighted homogeneous coordinates, we always use x, y, z, t, and w with $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, and $\mathrm{wt}(w) = a_4$.
- The number $\gimel$ always means the entry number of each family of weighted Fano hypersurfaces in The Table of Part 6.
- In each family, we always let X be a general quasismooth hypersurface of degree d in the weighted projective space $\mathbb{P}(1, a_1, a_2, a_3, a_4)$, where $d = \sum_{i=1}^4 a_i$.
- On the threefold X, a given Halphen pencil is denoted by $\mathcal{M}$.
- For a given Halphen pencil $\mathcal{M}$, we always assume that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$.
- When a morphism $f : V \to W$ is given, the proper transforms of a curve Z, a surface D, and a linear system $\mathcal{D}$ on W by the morphism f will be always denoted by Z_V, D_V, and $\mathcal{D}_V$, respectively, i.e., we use the ambient space V as their subscripts.
- S : the surface on X defined by the equation $x = 0$.
- S^y : the surface on X defined by the equation $y = 0$.
- S^z : the surface on X defined by the equation $z = 0$.
- S^t : the surface on X defined by the equation $t = 0$.
- S^w : the surface on X defined by the equation $w = 0$.
- C : the curve on X defined by the equations $x = y = 0$.
- $\bar{C}$: the curve on X defined by the equations $x = z = 0$.
- $\tilde{C}$: the curve on X defined by the equations $x = t = 0$.
- $\hat{C}$: the curve on X defined by the equations $x = w = 0$.

[4]The cases $\gimel = 18$, 22, 28 is not covered by the article [4]. Moreover, it has a mistake in Lemma 3.1. So we reprove Lemma 3.1 of [4] in this article. See K3-Propositions 4.1.1, 4.1.4, 4.2.2, and 4.3.2.

0.5. Set-up.

In each case, for a given general hypersurface X and a given Halphen pencil $\mathcal{M}$, we consider the log pair $(X, \frac{1}{n}\mathcal{M})$. Note that the natural number n is given by $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$. At the beginning of each section, we state the degree of X, the weights of the ambient weighted projective space, and the singularities of X. When the section contains a single case, we describe all the singular points. Because we could not put the values of b and c in Proposition 0.3.9, reader should refer to The Table in Part 6 for each singular point.

After we describe these simple features, we present elliptic fibrations into which the general hypersurface X can be birationally transformed. For detail, reader should refer to [3]. Note that for the cases $\daleth = 3$, 60, 75, 84, 87, 93, the hypersurface X cannot be birationally transformed into an elliptic fibration ([4]).

We try to show that the Halphen pencil $\mathcal{M}$ is one of the pencils given in Examples I, II, III, IV, and V. To do so, we will consider the set

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big).$$

Theorem 0.2.4 implies that *the set is always non-empty* since the linear system $\mathcal{M}$ is a Halphen pencil and the anticanonical divisor $-K_X$ is nef and big. Furthermore, due to Theorem 0.3.1, there is a birational automorphism $\rho \in \mathrm{Bir}(X)$ such that the log pair $(X, \frac{1}{\bar{n}}\rho(\mathcal{M}))$ is canonical for the natural number $\bar{n}$ with $\rho(\mathcal{M}) \sim_{\mathbb{Q}} -\bar{n}K_X$. It will turn out that the pencil $\rho(\mathcal{M})$ is one of the pencils constructed in Examples I, II, III, IV, and V that are $\mathrm{Bir}(X)$-invariant (Proposition 0.3.2). It implies $\mathcal{M} = \rho(\mathcal{M})$. For this reason, we may always assume that *the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical.*

At the first stage, we use Corollaries 0.3.7 and 0.3.8, if applicable, in order to exclude the case when the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve. Then, *usually the set contains only singular points of X due to Lemma 0.3.3.*

Next, we apply Lemmas 0.3.10 and 0.3.11 with The Table in order to minimize, as much as possible, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ to be considered.

After such things to do, we start the game to identify the Halphen pencil $\mathcal{M}$. We need to calculate base curves of proper transforms of various linear systems by various Kawamata blow ups, their multiplicities, and so on. Due to huge volume of calculations, we usually omit the calculations. However, from time to time, we present them to show how to calculate.

In addition, *unless otherwise mentioning, whenever we consider a log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ that is obtained from the log pair $(X, \frac{1}{n}\mathcal{M}_X)$ by a sequence of Kawamata blow ups, we always assume that $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$.* This condition is always satisfied when the Kawamata blow ups are obtained from centers of canonical singularities due to Lemma 0.2.6.

In Parts 3 and 4, we also show that general surfaces of Halphen pencils of types III, IV, and V are birational to smooth K3 surfaces. It will be proved directly or by using Corollary 0.2.12. Such statements are titled by K3-Proposition to be simply distinguished from the other works. The cases in Examples I and II are covered by Proposition 0.3.12.

Part 1. Fano threefold hypersurfaces with a single Halphen pencil $|-K_X|$.

1.1. Case $\gimel = 7$, hypersurface of degree 8 in $\mathbb{P}(1,1,2,2,3)$.

The threefold X is a general hypersurface of degree 8 in $\mathbb{P}(1,1,2,2,3)$ with $-K_X^3 = \frac{2}{3}$. The singularities of the hypersurface X consist of four points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point Q that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.

For each point P_i, there is a commutative diagram

$$\begin{tikzcd} U_i \arrow[d, "\alpha_i"'] & & Y_i \arrow[ll, "\beta_i"'] \arrow[d, "\eta_i"] \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where

- ξ_i is a rational map defined in the outside of the points P_i and Q,
- α_i is the Kawamata blow up at the point P_i with weights $(1,1,1)$,
- β_i is the Kawamata blow up with weights $(1,1,2)$ at the point Q_i whose image to X is the point Q,
- η_i is an elliptic fibration.

There is another rational map $\xi_0 : X \dashrightarrow \mathbb{P}(1,1,2)$ that gives us the following commutative diagram:

$$\begin{tikzcd} U_0 \arrow[d, "\alpha_0"'] & & Y_0 \arrow[ll, "\beta_0"'] \arrow[d, "\eta_0"] \\ X \arrow[rr, dashed, "\xi_0"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where

- α_0 is the Kawamata blow up at the point Q with weights $(1,1,2)$,
- β_0 is the Kawamata blow up with weights $(1,1,1)$ at the singular point O in the exceptional divisor of α_0,
- η_0 is an elliptic fibration.

The pencil $|-K_X|$ is invariant under the action of the group of birational automorphisms $\mathrm{Bir}(X)$ and hence we may assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P_1, P_2, P_3, P_4, Q\Big\}$$

due to Lemmas 0.3.3, 0.3.10, 0.3.11 and Corollary 0.3.8. Note that the base locus of the pencil $|-K_X|$ consists of the irreducible curve C defined by $x = y = 0$.

Lemma 1.1.1. *If the $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains distinct points P_i and P_j, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\pi_j : W \to U_i$ be the Kawamata blow up with weights $(1,1,1)$ at the point whose image to X is the point P_j and D be a general surface of the pencil $|-K_X|$. The base locus of the pencil $|-K_W|$ consists of the irreducible curve C_W. The surface D_W is normal and $C_W^2 = -\frac{1}{3}$ on the surface D_W, which implies that $\mathcal{M}_W = |-K_W|$ by Theorem 0.2.9. □

The singularities of the log pair $(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ are not terminal because the divisor $-K_{U_i}$ is nef and big.

Lemma 1.1.2. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot consist of a single point P_i.*

Proof. Suppose that the singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ are terminal in the outside the singular point P_i. Let E_i be the exceptional divisor of α_i. Then, the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ contains a line $Z \subset E_i \cong \mathbb{P}^2$ by Lemmas 0.2.6 and 0.2.7. But this is a contradiction because of Lemma 0.2.3. □

Lemma 1.1.3. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the points P_i and Q, then $\mathcal{M} = |-K_X|$.*

Proof. Let G_i be the exceptional divisor of β_i. Then, $\mathcal{M}_{Y_i} \sim_{\mathbb{Q}} -nK_{Y_i}$ by Lemma 0.2.6, which implies that every member of the pencil $\mathcal{M}_{Y_i}$ is contracted to a curve by the morphism η_i. In particular, the base locus of $\mathcal{M}_{Y_i}$ does not contain curves that are not contracted by η_i. Therefore, the set $\mathbb{CS}(Y_i, \frac{1}{n}\mathcal{M}_{Y_i})$ contains the singular point O_i of the surface $G_i \cong \mathbb{P}(1,1,2)$ due to Theorem 0.2.4 and Lemma 0.2.7.

Let $\pi_i : W_i \to Y_i$ be the Kawamata blow up at the point O_i with weights $(1,1,1)$ and D be a general surface of the pencil $|-K_X|$. The base locus of the pencil $|-K_{W_i}|$ consists of the irreducible curve C_{W_i}. The surface D_{W_i} is normal and $C_{W_i}^2 = -\frac{1}{2}$ on the surface D_{W_i}, which implies that $\mathcal{M}_{W_i} = |-K_{W_i}|$ by Theorem 0.2.9. □

Lemma 1.1.4. *The log pair $(U_0, \frac{1}{n}\mathcal{M}_{U_0})$ is terminal in the outside of the singular point O of the exceptional divisor of the birational morphism β_0.*

Proof. Let E_0 be the exceptional divisor of the birational morphism β_0. Suppose that the log pair $(U_0, \frac{1}{n}\mathcal{M}_{U_0})$ is not terminal in the outside of the singular point O. Then, the set $\mathbb{CS}(U_0, \frac{1}{n}\mathcal{M}_{U_0})$ consists of the point O and an irreducible curve $L \subset E_0 \cong \mathbb{P}(1,1,2)$ such that $L \in |\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$.

The threefold X can be given by the equation

$$w^2 z + w f_5(x,y,z,t) + f_8(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Moreover, we may assume that the curve L is cut out on the surface E_0 by the surface S_{U_0}.

Let $\mathcal{P}$ be the pencil on X cut out on the threefold X by the pencil $\lambda x^2 + \mu z = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. Then, the base locus of $\mathcal{P}$ consists of the irreducible curve $\bar{C}$ cut by the equations $x = z = 0$.

The base locus of the linear system $\mathcal{P}_{U_0}$ consists of the irreducible curves L and $\bar{C}_{U_0}$. A general surface D in $\mathcal{P}_{U_0}$ is normal. Moreover, we have

$$\mathcal{M}_{U_0}\Big|_D \equiv -nK_{U_0}\Big|_D \equiv nS_{U_0}\Big|_D \equiv n\big(L + \bar{C}_{U_0}\big)$$

by Lemma 0.2.6. On the other hand, we have $S_{U_0} \cdot D = L + \bar{C}_{U_0}$ and $E_0 \cdot D = 2L$, which implies that $\bar{C}_{U_0}^2 = -\frac{5}{4}$ on the surface D. The latter together with the equality $\mathrm{mult}_L(\mathcal{M}_{U_0}) = n$ easily implies that $\mathcal{M}_{U_0} = \mathcal{P}_{U_0}$ due to Theorem 0.2.9. We obtain $n = 2$, but D is normal and $\mathrm{mult}_L(D) \neq 2$, which is a contradiction. □

Proposition 1.1.5. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. By the previous arguments, we have only to consider the case when

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{Q\Big\}$$

Then, $\mathcal{M}_{Y_0} \sim_{\mathbb{Q}} -nK_{Y_0}$, which implies that every member of the pencil $\mathcal{M}_{Y_0}$ is contracted to a curve by the morphism η_0. In particular, the base locus of the pencil $\mathcal{M}_{Y_0}$ does not contain any curves by Lemma 1.1.4. Thus, the singularities of the log pair $(Y_0, \frac{1}{n}\mathcal{M}_{Y_0})$ are terminal by Lemma 0.2.7, which is impossible by Theorem 0.2.4. □

1.2. Cases $\beth = 9$, 11, and 30.

We first consider the case $\beth = 9$. The variety X is a general hypersurface of degree 9 in $\mathbb{P}(1,1,2,3,3)$ with $-K_X^3 = \frac{1}{2}$. The singularities of the hypersurface X consist of one point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ and three points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1,1,2)$.

We have the following commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the composition of the Kawamata blow ups at the points P_1, P_2, P_3 with weights $(1,1,2)$.
- η is an elliptic fibration.

There is another elliptic fibration as follows:

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "\eta_0"] & \\ X \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where

- π is the Kawamata blow up at the point O with weights $(1,1,1)$,
- η_0 is an elliptic fibration.

It follows from Lemma 0.3.3 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain smooth points of the hypersurface X. Moreover, if it contains a curve then we obtain $\mathcal{M} = |-K_X|$ from Corollary 0.3.8. Therefore, we may assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P_1, P_2, P_3, O\Big\}.$$

Lemma 1.2.1. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not consist of the point O.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point O. Then, $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$ by Lemma 0.2.6, which implies that every member in the pencil $\mathcal{M}_V$ is contracted to a curve by the morphism η_0. In particular, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ does not contain curves. On the other hand, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ is not empty by Theorem 0.2.4. Hence, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains a point of the exceptional divisor of π, which is impossible by Lemma 0.2.7. $\square$

Note that the base locus of $|-K_X|$ consists of the irreducible curve C cut by $x = y = 0$.

Lemma 1.2.2. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the points O and P_i, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\beta_i : W_i \to V$ be the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P_i. Then, $|-K_{W_i}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of $|-K_{W_i}|$ consists of the irreducible curve C_{W_i}. One can easily see that $-K_{W_i} \cdot C_{W_i} < 0$. On the other hand, we have $\mathcal{M}_{W_i} \sim_{\mathbb{Q}} -nK_{W_i}$ by Lemma 0.2.6, which implies that $\mathcal{M}_{W_i} = |-K_{W_i}|$ by Theorem 0.2.9. $\square$

Therefore, we may further assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P_1, P_2, P_3\Big\}.$$

Let $\alpha_i : U_i \to X$ be the Kawamata blow up at the point P_i and F_i be the exceptional divisor of α_i. The exceptional divisor F_i contains a singular point Q_i that is a quotient singular point of the threefold U_i of type $\frac{1}{2}(1,1,1)$.

Lemma 1.2.3. *If the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ contains the point Q_i, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\beta_i : Y_i \to U_i$ be the Kawamata blow up at the point Q_i. Then, $|-K_{Y_i}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of $|-K_{V_i}|$ consists of the irreducible curve C_{Y_i}. Also, we have $-K_{Y_i} \cdot C < 0$. On the other hand, $\mathcal{M}_{Y_i} \sim_{\mathbb{Q}} -nK_{Y_i}$ by Lemma 0.2.6, which implies that $\mathcal{M}_{Y_i} = |-K_{Y_i}|$ by Theorem 0.2.9. Hence, we obtain $\mathcal{M} = |-K_X|$. $\square$

Thus, it follows from Lemma 0.2.7 that we may assume that the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ does not contain subvarieties of F_i in the case when the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P_i.

Lemma 1.2.4. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not consist of the point P_i.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P_i. Then, $\mathcal{M}_{U_i} \sim_{\mathbb{Q}} -nK_{U_i}$ by Lemma 0.2.6, which implies that the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ is not empty by Theorem 0.2.4, because the divisor $-K_{U_i}$ is nef and big. Therefore, the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ must contain a subvariety of G_i, which is impossible by our assumption. $\square$

Proposition 1.2.5. *If $\gimel = 9$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. By the proof of Lemma 1.2.4, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P_1, P_2, P_3\}$, which implies that $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 0.2.6. Therefore, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must contain a subvariety of an exceptional divisor of α by Theorem 0.2.4, which is impossible, because we assumed that the set $\mathbb{CS}(U_i, \frac{1}{n}\mathcal{M}_{U_i})$ does not contain subvarieties of F_i. The obtained contradiction concludes the proof. $\square$

For the case $\gimel = 30$, let X be a general hypersurface of degree 16 in $\mathbb{P}(1,1,3,4,8)$ with $-K_X^3 = \frac{1}{6}$. The singularities of X consist of one point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and two points P_1, P_2 that are quotient singularities of type $\frac{1}{4}(1,1,3)$.

We have the following commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the composition of the Kawamata blow ups at the points P_1 and P_2 with weights $(1,1,3)$.
- η is an elliptic fibration.

There is another elliptic fibration as follows:

$$\begin{tikzcd} & V \arrow[dl, "\pi"'] \arrow[dr, "\eta_0"] & \\ X \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where

- π is the Kawamata blow up at the point O with weights $(1,1,2)$,
- η_0 is an elliptic fibration.

Proposition 1.2.6. *If $\gimel = 30$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. The proof is the same as the case $\gimel = 9$. $\square$

In the case $\gimel = 11$, the threefold X is a general hypersurface of degree 10 in $\mathbb{P}(1,1,2,2,5)$ with $-K_X^3 = \frac{1}{2}$. Its singularities consist of five points $P_1, \cdots, P_5$ that are quotient singularities

of type $\frac{1}{2}(1,1,1)$. For each singular point P_i, we have an elliptic fibration as follows:

$$\begin{tikzcd}
& U_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\
X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where

- π_i is the Kawamata blow up at the point P_i with weights $(1,1,1)$,
- η_i is an elliptic fibration.

Proposition 1.2.7. *If $\gimel = 11$, the linear system $|-K_X|$ is a unique Halphen pencil on X.*

Proof. If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then we obtain $\mathcal{M} = |-K_X|$ from Corollary 0.3.8. Thus, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P_1, P_2, P_3, P_4, P_5\Big\}$$

by Lemma 0.3.3. Furthermore, it cannot consist of a single point by Lemmas 0.2.3 and 0.2.7. Therefore, it contains at least two, say P_i and P_j, of the five singular points. Let $\pi : U \to U_i$ be the Kawamata blow up at the singular point whose image to X is the point P_j. Then, the pencil $|-K_U|$ is the proper transform of the pencil $|-K_X|$ and its base locus consists of the irreducible curve C_U. Because $-K_U \cdot C_U < 0$ and $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$, Theorem 0.2.9 completes the proof. $\square$

1.3. Case $\gimel = 12$, hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$.

The threefold X is a general hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$ with $-K_X^3 = \frac{5}{12}$. The singularities of the hypersurface X consist of two singular points that are quotient singularities of type $\frac{1}{2}(1,1,1)$, one point P that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and one point Q that is a quotient singularity of type $\frac{1}{4}(1,1,3)$.

We have the following commutative diagram:

$$\begin{tikzcd}
& & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drr, "\eta"] & & \\
& U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & \mathbb{P}(1,1,2), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & \\
& X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,2)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,1,3)$,
- β_Q is the Kawamata blow up with weights $(1,1,3)$ at the point whose image by the birational morphism α_P is the point Q,
- β_P is the Kawamata blow up with weights $(1,1,2)$ at the point whose image by the birational morphism α_Q is the point P,
- β_O is the Kawamata blow up with weights $(1,1,2)$ at the singular point O of the variety U_Q that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the birational morphism α_Q,

- γ_P is the Kawamata blow up with weights $(1,1,2)$ at the point whose image by the birational morphism $\alpha_Q \circ \beta_O$ is the point P,
- γ_O is the Kawamata blow up with weights $(1,1,2)$ at the singular point of the variety U_{PQ} that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2z + f_6(x,y,z,t)w + f_{10}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & U_Q \arrow[r, "\alpha_Q"] \arrow[dl, "\pi"'] & X \arrow[dl, dashed, "\chi"'] \arrow[dr, dashed, "\psi"] & \\
W \arrow[r, "\omega"'] & \mathbb{P}(1,1,2,3) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where

- ξ and χ are the natural projections,
- π is a birational morphism,
- ω is a double cover of $\mathbb{P}(1,1,2,3)$ ramified along a surface R of degree 12.

The surface R is given by the equation

$$f_6(x,y,z,t)^2 - 4zf_{10}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,3) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big),$$

which implies that the surface R has exactly 20 ordinary double points given by the equations $z = f_6 = f_{10} = 0$. Thus, the morphism π contracts 20 smooth rational curves $C_1, \cdots, C_{20}$ to isolated ordinary double points of the variety W that dominate the singular points of R in $\mathbb{P}(1,1,2,3)$.

Proposition 1.3.1. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

Suppose that $\mathcal{M} \neq |-K_X|$. Let us show that this assumption leads us to a contradiction.

It follows from Corollary 0.3.8 and Lemmas 0.3.3, 0.3.11 that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}.$$

Lemma 1.3.2. *The set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ cannot contain a curve.*

Proof. Suppose that the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains an irreducible curve Z. Let G be the exceptional divisor of the birational morphism α_Q. Then, $G \cong \mathbb{P}(1,1,3)$ and $Z \subset G$. Moreover, it follows from Lemma 0.2.7 that Z is a curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,1,3)}(1)|$ on the surface G.

We consider the surface S^z on X. Let Z' be the curve $S^z_{U_Q} \cap G$. Then, the surface $S^z_{U_Q}$ contains every curve C_i, but not the curve Z, and its image S^z_W by the morphism π is isomorphic to $\mathbb{P}(1,1,3)$. The curve Z' is smooth and $\alpha_Q|_{Z'}$ is a double cover.

Because the hypersurface X is general, the surface $S^z_{U_Q}$ is smooth along the curves C_i, the morphism $\pi|_{S^z_{U_Q}}$ contracts the curve C_i to a smooth point of S^z_W, and either the intersection $Z \cap Z'$ consists of two points or the point $Z \cap Z'$ is not contained in $\cup_{i=1}^{20} C_i$.

For general surfaces M_{U_Q}, M'_{U_Q} in $\mathcal{M}_{U_Q}$ and a general surface D in $|-6K_{U_Q}|$, we have

$$2n^2 = D \cdot M_{U_Q} \cdot M'_{U_Q} \geq 2\mathrm{mult}_Z(M_{U_Q} \cdot M'_{U_Q}) \geq 2\mathrm{mult}_Z(M_{U_Q})\mathrm{mult}_Z(M'_{U_Q}) \geq 2n^2,$$

which immediately implies that the support of the cycle $M_{U_Q} \cdot M'_{U_Q}$ is contained in the union of the curve L and $\cup_{i=1}^{20} C_i$. Hence, we have

$$\mathcal{M}_{U_Q}\Big|_{S^z_{U_Q}} = \mathcal{D} + \sum_{i=1}^{20} m_i C_i,$$

where m_i is a natural number and $\mathcal{D}$ is a pencil without fixed components.

Let P' be a point of $Z \cap Z'$ and D_1, D_2 be general curves in $\mathcal{D}$. Then,

$$\mathrm{mult}_{P'}(D_1) = \mathrm{mult}_{P'}(D_2) \geq \begin{cases} n & \text{in the case when } P' \notin \cup_{i=1}^{20} C_i, \\ n - m_i & \text{in the case when } P' \in C_i, \end{cases}$$

which implies that

$$\frac{n^2}{3} - \sum_{i=1}^{20} m_i^2 = D_1 \cdot D_2 \geq \sum_{P' \in Z \cap Z'} \mathrm{mult}_{P'}(D_1)\mathrm{mult}_{P'}(D_2).$$

However, it is impossible because $m_i^2 + (n - m_i)^2 \geq \frac{n^2}{4}$. □

The exceptional divisor E_P of α_P contains a singular point P_1 of U_P that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

Lemma 1.3.3. *The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ cannot contain the singular point P_1.*

Proof. Let $\sigma_P : V_P \to U_P$ be the Kawamata blow up at the singular point P_1 and $\mathcal{P}$ be the proper transform of the pencil $|-K_X|$ via the birational morphism $\alpha_P \circ \sigma_P$. Then, $\mathcal{M}_{V_P} \sim_{\mathbb{Q}} -nK_{V_P}$ by Lemma 0.2.6 and

$$\mathcal{P} \sim_{\mathbb{Q}} -K_{V_P} \sim_{\mathbb{Q}} (\alpha_P \circ \sigma_P)^*(-K_X) - \frac{1}{3}\sigma_P^*(E_P) - \frac{1}{2}F_P,$$

where F_P is the exceptional divisor of σ_P. Also, it has a unique base curve C_{V_P}. On a general surface $S_{V_P} \in \mathcal{P}$, the self-intersection number $C^2_{V_P} = -K^3_{V_P}$ is negative. Also we have

$$\mathcal{M}_{V_P}\Big|_{S_{V_P}} \equiv -nK_{V_P}\Big|_{S_{V_P}} \equiv nC_{V_P},$$

which implies that $\mathcal{M}$ is the pencil $|-K_X|$ by Theorem 0.2.9. However, we assumed that $\mathcal{M} \neq |-K_X|$. □

Therefore, the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ must consist of the point $\bar{Q}$ whose image to X is the point Q because it is not empty by Theorem 0.2.4 and it cannot contain a curve by Lemma 0.2.3.

Meanwhile, the exceptional divisor of β_O contains a singular point O_1 of U_{QO} that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

Lemma 1.3.4. *The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ cannot contain the point O_1.*

Proof. Let $\sigma_O : V_O \to U_{QO}$ be the Kawamata blow up at the point O_1. Then, the pencil $|-K_{V_O}|$ is the proper transform the pencil $|-K_X|$. Its base locus consists of the irreducible curve C_{V_O}. Because $\mathcal{M}_{V_O} \sim_{\mathbb{Q}} -nK_{V_O}$ and $-K_{V_O} \cdot C_{V_O} < 0$, we obtain from Theorem 0.2.9 that $\mathcal{M} = |-K_X|$, which contradicts our assumption. □

By the previous lemmas, we can see $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P, Q\}$. Furthermore, the set $\mathbb{CS}(U_{PQ}, \frac{1}{n}\mathcal{M}_{U_{PQ}})$ consists of the singular point of U_{PQ} contained in the exceptional divisor β_Q. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must contain the singular point contained in the exceptional divisor γ_O. In such a case, Lemma 1.3.4 shows a contradiction $\mathcal{M} = |-K_X|$.

1.4. Case $\daleth = 13$, hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$.

The threefold X is a general hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$ with $-K_X^3 = \frac{11}{30}$. It has three singular points. One is a quotient singularity of type $\frac{1}{2}(1,1,1)$, another is a quotient singular point P of type $\frac{1}{3}(1,1,2)$, and the other is a quotient singular point Q of type $\frac{1}{5}(1,2,3)$.

We have the following commutative diagram:

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drr, "\eta"] & & \\
 & U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & \mathbb{P}(1,1,2), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,2)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,2,3)$,
- β_Q is the Kawamata blow up with weights $(1,2,3)$ at the point whose image by the birational morphism α_P is the point Q,
- β_P is the Kawamata blow up with weights $(1,1,2)$ at the point whose image by the birational morphism α_Q is the point P,
- β_O is the Kawamata blow up with weights $(1,2,1)$ at the singular point O of the variety U_Q that is a quotient singularity of type $\frac{1}{3}(1,2,1)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,1,2)$ at the point whose image by the birational morphism $\alpha_Q \circ \beta_O$ is the point P,
- γ_O is the Kawamata blow up with weights $(1,2,1)$ at the singular point of the variety U_{PQ} that is a quotient singularity of type $\frac{1}{3}(1,2,1)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

Note that the base locus of the pencil $|-K_X|$ consists of the irreducible curve C defined by $x = y = 0$.

It follows from Corollary 0.3.8 and Lemmas 0.3.3, 0.3.11 we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subset \{P, Q\}$.

Lemma 1.4.1. *If $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$, then $\mathcal{M} = |-K_X|$.*

Proof. It follows from Theorem 0.2.4 that the log pair $(U_P, \mathcal{M}_{U_P})$ is not terminal at the singular point P_1 contained in the exceptional divisor of α_P.

Let $\beta_1 : W_P \to U_P$ be the Kawamata blow up at the singular point P_1 with weights $(1,1,1)$. Then, the pencil $|-K_{W_P}|$ is the proper transform of $|-K_X|$. It has a unique irreducible base curve C_{W_P}. We have $-K_{W_P} \cdot C_{W_P} = -K_{W_P}^3 = -\frac{3}{10}$ and $\mathcal{M}_{W_P} \sim_{\mathbb{Q}} -nK_{W_P}$. Therefore, we obtain $\mathcal{M} = |-K_X|$ from Theorem 0.2.9. □

The exceptional divisor $E \cong \mathbb{P}(1,2,3)$ of the birational morphism α_Q contains two singular points O and O_2 of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{2}(1,1,1)$, respectively. We denote the unique irreducible curve in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on E by L.

Lemma 1.4.2. *If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains the point O_2, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\beta_2 : W_Q \to U_Q$ be the Kawamata blow up at the point O_2 with weights $(1,1,1)$. The pencil $|-K_{W_Q}|$ is the proper transform of $|-K_X|$. It has exactly two irreducible base curves C_{W_Q} and L_{W_Q}. On a general surface D in the pencil $|-K_{W_Q}|$, we have

$$L_{W_Q}^2 = -\frac{4}{3},\ C_{W_Q}^2 = -\frac{5}{6},\ L_{W_Q}\cdot C_{W_Q} = 1.$$

The surface D is normal. The curves C_{W_Q} and L_{W_Q} form a negative-definite intersection form on D. On the other hand, $\mathcal{M}_{W_Q}|_D \equiv nC_{W_Q} + nL_{W_Q}$ by Lemma 0.2.6. Therefore, we obtain $\mathcal{M} = |-K_X|$ by Theorem 0.2.9. ☐

Lemma 1.4.3. *If the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the singular point O_3 contained in the exceptional of β_O, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\gamma_1 : W_{QO} \to U_{QO}$ be the Kawamata blow up at the point O_3 with weights $(1,1,1)$. Then, the pencil $|-K_{W_{QO}}|$ is the proper transform of the pencil $|-K_X|$. It has exactly two irreducible base curves $C_{W_{QO}}$ and $L_{W_{QO}}$. On a general surface D in $|-K_{W_{QO}}|$, we have

$$L_{W_{QO}}^2 = -\frac{3}{2},\ C_{W_{QO}}^2 = -\frac{5}{6},\ L_{W_{QO}}\cdot C_{W_{QO}} = 1.$$

The surface D is normal. On the other hand, $\mathcal{M}_{W_{QO}}|_D \equiv nC_{W_{QO}} + nL_{W_{QO}}$ by Lemma 0.2.6. Since the curves $C_{W_{QO}}$ and $L_{W_{QO}}$ form a negative-definite intersection form on the normal surface D, it follows from Theorem 0.2.9 that $\mathcal{M} = |-K_X|$. ☐

Proposition 1.4.4. *If $\gimel = 13$, then $\mathcal{M} = |-K_X|$.*

Proof. Due to the previous lemmas, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{P, Q\Big\}.$$

Following the weighted blow ups $Y \to U_{QO} \to U_Q \to X$ and using Lemmas 1.4.2 and 1.4.3, we can furthermore assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point contained the exceptional divisor of the birational morphism γ_P. The statement then follows from the proof Lemma 1.4.1. ☐

1.5. Cases $\gimel = 15$, 17, and 41.

We are to prove the following:

Proposition 1.5.1. *If $\gimel = 15$, 17, or 41, then the linear system $|-K_X|$ is a unique Halphen pencil on X.*

We consider the case of $\gimel = 15$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 12 in $\mathbb{P}(1,1,2,3,6)$ with $-K_X^3 = \frac{1}{3}$. Then, the singularities of X consist of two singular points P and Q that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and two points of type $\frac{1}{2}(1,1,1)$. We have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\sigma_Q"'] \arrow[dr, "\sigma_P"] \arrow[drrr, "\eta"] & & & \\
Y_P \arrow[dr, "\pi_P"'] & & Y_Q \arrow[dl, "\pi_Q"] & & \mathbb{P}(1,1,2), \\
 & X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,1,2)$,
- π_Q is the Kawamata blow up at Q with weights $(1,1,2)$,

- σ_Q is the Kawamata blow up with weights $(1,1,2)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,1,2)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ is nonempty by Theorem 0.2.4. If it contains either a singular point of type $\frac{1}{2}(1,1,1)$ on X or a curve, then the identity $\mathcal{M} = |-K_X|$ follows from Lemma 0.3.11 and Corollary 0.3.8, respectively. Furthermore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}$$

due to Lemma 0.3.3. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P. The exceptional divisor E_P of π_P contains a singular point O of type $\frac{1}{2}(1,1,1)$.

Lemma 1.5.2. *If the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the singular point O, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\alpha : W \to Y_P$ be the Kawamata blow up at the point O with weights $(1,1,1)$. The linear system $|-K_W|$ is the proper transform of the linear system $|-K_X|$. It has a single base curve C_W. On a general surface D_W in $|-K_W|$, we have $C_W^2 = -\frac{1}{3}$. Note that the surface D_W is normal. Meanwhile, we have

$$\mathcal{M}_W\Big|_{D_W} \equiv -nK_W\Big|_{D_W} \equiv nC_W.$$

Therefore, it follows from Theorem 0.2.9 that the pencil $\mathcal{M}$ coincides with $|-K_X|$. □

Therefore, we may assume that the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ consists of the single point Q_1. Then, we may assume that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the singular point of type $\frac{1}{2}(1,1,1)$ that is contained in the exceptional divisor of σ_Q. We can then show $\mathcal{M} = |-K_X|$ as in the proof of Lemma 1.5.2.

With the exactly same way as above, we can show that $\mathcal{M} = |-K_X|$ if the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point Q.

Now, we consider the case of $\daleth = 41$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 20 in $\mathbb{P}(1,1,4,5,10)$ with $-K_X^3 = \frac{1}{10}$. Then, it has two singular points P and Q that are quotient singularities of type $\frac{1}{5}(1,1,4)$ and two singular points of type $\frac{1}{2}(1,1,1)$.

We have a commutative diagram

$$\begin{tikzcd}
& V \arrow[dl, "\sigma_Q"'] \arrow[dr, "\sigma_P"] \arrow[drrr, "\eta"] & & & \\
Y_P \arrow[dr, "\pi_P"'] & & Y_Q \arrow[dl, "\pi_Q"] & & \mathbb{P}(1,1,4), \\
& X \arrow[urrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,1,4)$,
- π_Q is the Kawamata blow up at Q with weights $(1,1,4)$,
- σ_Q is the Kawamata blow up with weights $(1,1,4)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,1,4)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

Using the exactly same method as in the case $\daleth = 15$, one can show that $\mathcal{M} = |-K_X|$.

For the case $\daleth = 17$, let X be the hypersurface given by a general quasihomogeneous equation of degree 12 in $\mathbb{P}(1,1,3,4,4)$ with $-K_X^3 = \frac{1}{4}$. Then, it has three singular points P_1, P_2, and P_3 that are quotient singularities of type $\frac{1}{4}(1,1,3)$.

In this case, we have the following commutative diagram:

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the points P_1, P_2, and P_3 with weights $(1,1,3)$,
- η is and elliptic fibration.

Even though three singular points are involved in this case, the same method as in the previous cases can be applied to obtain $\mathcal{M} = |-K_X|$.

1.6. Case $\daleth = 16$, hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$.

The threefold X is a general hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$ with $-K_X^3 = \frac{3}{10}$. Its singularities consist of three quotient singularities of type $\frac{1}{2}(1,1,1)$ and one point O that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2) \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,1,4)$,
- β is the Kawamata blow up with weights $(1,1,3)$ at the singular point of the variety U that is contained in the exceptional divisor of α,
- γ is the Kawamata blow up with weights $(1,1,2)$ at the singular point of W that is contained in the exceptional divisor of β,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2z + f_7(x,y,z,t)w + f_{12}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Moreover, there is commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[r, "\alpha"] & X \arrow[dl, dashed, "\chi"'] \arrow[dr, dashed, "\psi"] & \\ V \arrow[r, "\omega"'] & \mathbb{P}(1,1,2,4) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,2), \end{tikzcd}$$

where

- ξ and χ are the natural projections,
- π is a birational morphism,

- ω is a double cover of $\mathbb{P}(1,1,2,4)$ ramified along a surface R of degree 12.

The surface R is given by the equation

$$f_7(x,y,z,t)^2 - 4zf_{12}(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,4) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big),$$

which implies that R has 21 isolated ordinary double points, given by the equations $z = f_7 = f_{12} = 0$. The morphism π contracts 21 smooth rational curves $C_1, C_2, \cdots, C_{21}$ to isolated ordinary double points of V which dominate the singular points of R.

Proposition 1.6.1. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

To prove Proposition 1.6.1, due to Corollary 0.3.8 and Lemmas 0.3.3 and 0.3.11, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}.$$

Let E be the exceptional divisor of the birational morphism α. It contains one singular point P of U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. The surface E is isomorphic to $\mathbb{P}(1,1,4)$. The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P by Theorem 0.2.4 and Lemma 0.2.7. Furthermore, the following shows it consists of the point P.

Lemma 1.6.2. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain a curve.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains a curve Z. Then, Z is contained in the surface E. Furthermore, it follows from Lemma 0.2.7 that Z is a curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,1,4)}(1)|$. Therefore, for a general surface M in $\mathcal{M}$, we have

$$\mathrm{Supp}\Big(M_U \cdot E\Big) = Z$$

because $M_U|_E \sim_{\mathbb{Q}} |\mathcal{O}_{\mathbb{P}(1,1,4)}(n)|$ and $\mathrm{mult}_Z(M_U) \geq n$.

Let M'_U be a general surface in $\mathcal{M}_U$ and D be a general surface in $|-4K_U|$. Then,

$$n^2 = D \cdot M_U \cdot M'_U \geq \mathrm{mult}_Z(M_U \cdot M'_U) \geq \mathrm{mult}_Z(M_U)\mathrm{mult}_Z(M'_U) \geq n^2,$$

which implies that $\mathrm{mult}_L(M_U \cdot M'_U) = n^2$ and

$$\mathrm{Supp}\Big(M_U \cdot M'_U\Big) \subset Z \cup \bigcup_{i=1}^{21} C_i.$$

We consider the surface S^z. The image S^z_V of S^z_U to V is isomorphic to $\mathbb{P}(1,1,4)$. The surface S^z_U does not contain the curve Z due to the generality in the choice of X, but it contains every curve C_i. Moreover, the surface S^z_U is smooth along the curves C_i and the morphism $\pi|_{S^z_U}$ contracts the curve C_i to a smooth point of S^z_V. Hence, we have

$$\mathcal{M}_U\Big|_{S^z_U} = \mathcal{D} + \sum_{i=1}^{21} m_i C_i,$$

where m_i is a natural number and $\mathcal{D}$ is a pencil without fixed components. Therefore, the inequality $m_i > 0$ implies that $C_i \cap Z \neq \varnothing$ and there is a point P' of the intersection $Z \cap S^z_U$ that is different from the singular point P. We may assume that $m_1 > 0$. Let D_1 and D_2 be general curves in $\mathcal{D}$. Then,

$$\mathrm{mult}_{P'}(D_1) = \mathrm{mult}_{P'}(D_2) \geq \begin{cases} n & \text{in the case when } P' \notin \cup_{i=1}^{21} C_i, \\ n - m_i & \text{in the case when } P' \in C_i, \end{cases}$$

and the curves D_1 and D_2 pass through the point P because the point P is a base point of the pencil $\mathcal{M}_U$. Therefore, we have

$$\frac{n^2}{4} - \sum_{i=1}^{21} m_i^2 = D_1 \cdot D_2 > \mathrm{mult}_P(D_1)\mathrm{mult}_P(D_2) \geq (n - m_1)^2 \geq \frac{n^2}{4} - m_1^2,$$

which is a contradiction. $\square$

Let F be the exceptional divisor of the birational morphism β. It contains the singular point Q of W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$. The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the singular point Q by Theorem 0.2.4, Lemmas 0.2.7 and 0.2.3.

Let G be the exceptional divisor of γ and Q_1 be the unique singular point of G. The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must consist of the point Q_1 by Theorem 0.2.4 and Lemma 0.2.7 because every member in $\mathcal{M}_Y$ is contracted to a curve by the morphism η.

Let $\sigma : V_1 \to Y$ be the Kawamata blow up at the point Q_1. Then, $\mathcal{M}_{V_1} \sim_{\mathbb{Q}} -nK_{V_1}$ by Lemma 0.2.6, the linear system $|-K_{V_1}|$ is the proper transform of the pencil $|-K_X|$, and the base locus of the pencil $|-K_{V_1}|$ consist of the curve C_{V_1}. Therefore, the inequality $-K_{V_1} \cdot C_{V_1} < 0$ implies $\mathcal{M} = |-K_X|$ by Theorem 0.2.9.

1.7. Cases $\daleth = 20$ and 31.

First, we consider the case $\daleth = 20$. The threefold X is a general hypersurface of degree 13 in $\mathbb{P}(1,1,3,4,5)$ with $-K_X^3 = \frac{13}{60}$. It has three singular points. One is a quotient singularity P of type $\frac{1}{4}(1,1,3)$, another is a quotient singularity Q of type $\frac{1}{5}(1,1,4)$, and the other is a quotient singularity Q' of type $\frac{1}{3}(1,1,2)$.

There is a commutative diagram

$$\begin{tikzcd}
& & V \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drrrr, "\eta"] & & & & \\
& U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & & \mathbb{P}(1,1,3), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & & \\
& X \arrow[uurrrrr, dashed, "\psi"'] & & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,1,4)$,
- β_Q is the Kawamata blow up with weights $(1,1,4)$ at the point whose image to X is the point Q.
- β_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P
- β_O is the Kawamata blow up with weights $(1,1,3)$ at the singular point O of U_Q contained in the exceptional divisor of α_Q,
- γ_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P
- γ_O is the Kawamata blow up with weights $(1,1,3)$ at the singular point of U_{PQ} contained in the exceptional divisor of β_Q,
- η is an elliptic fibration.

There is a rational map $\xi : X \dashrightarrow \mathbb{P}(1,1,4)$ that gives us another commutative diagram

$$\begin{tikzcd}
& Y \arrow[dl, "\gamma_Q"'] \arrow[dr, "\gamma_{Q'}"] \arrow[drrrr, "\eta_0"] & & & & \\
U_{Q'} \arrow[dr, "\alpha_{Q'}"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \mathbb{P}(1,1,4), \\
& X \arrow[urrrr, dashed, "\xi"'] & & & &
\end{tikzcd}$$

where

- $\alpha_{Q'}$ is the Kawamata blow up at the point Q' with weights $(1,1,2)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,1,4)$,
- γ_Q is the Kawamata blow up with weights $(1,1,4)$ at the point whose image to X is the point Q,
- $\gamma_{Q'}$ is the Kawamata blow up with weights $(1,1,2)$ at the point whose image to X is the point Q',
- η_0 is an elliptic fibration.

It follows from Lemma 0.3.3 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain smooth points of the hypersurface X. Moreover, Corollary 0.3.8 implies that $\mathcal{M} = |-K_X|$ in the case when the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve. We assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain a curve, which implies that $\mathcal{M} \neq |-K_X|$. Then, $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subseteq \{P, Q, Q'\}$.

Lemma 1.7.1. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain both the points Q and Q'.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains both the points Q and Q'. Then, $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, which implies that every member in the pencil $\mathcal{M}_Y$ is contracted to a curve by the elliptic fibration η_0.

Let $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ be the singular points of Y whose image to X are the points P_1, P_2, P_3, respectively. Then,

$$\mathbb{CS}\Big(Y, \frac{1}{n}\mathcal{M}_Y\Big) \cap \Big\{\bar{P}_1, \bar{P}_2, \bar{P}_3\Big\} \neq \varnothing$$

by Theorem 0.2.4 and Lemma 0.2.7.

Let $\pi_Q : W_Q \to Y$ be the Kawamata blow up of the point $\bar{P}_i$ that is contained in the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ and D be a general surface in $|-K_{W_Q}|$. Then, $|-K_{W_Q}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_{W_Q}|$ consists of the irreducible curve C_{W_Q}. We can easily check $D \cdot C_{W_Q} < 0$. Hence, we obtain $\mathcal{M} = |-K_X|$ by Theorem 0.2.9 because $\mathcal{M}_{W_Q} \sim_{\mathbb{Q}} nD$ by Lemma 0.2.6. But it is impossible by our assumption. □

Lemma 1.7.2. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain both the points P and Q'.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains both the points P and Q'. Let $\pi_P : W_P \to U_P$ be the Kawamata blow up at the point whose image to X is the point Q'. Then, $|-K_{W_P}|$ is the proper transform of the pencil $|-K_X|$, the base locus of the pencil $|-K_{W_P}|$ consists of the irreducible curve C_{W_P}, and $D \cdot C_{W_P} = -\frac{1}{30}$. Hence, $\mathcal{M} = |-K_X|$ by Theorem 0.2.9 because $\mathcal{M}_{W_P} \sim_{\mathbb{Q}} nD$ by Lemma 0.2.6. But it is impossible by our assumption. □

Lemma 1.7.3. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot consist of a single point.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P. Then, $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ by Lemma 0.2.6. Hence, it follows from Theorem 0.2.4, Lemmas 0.2.7 and 0.2.3 that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ consists of the singular point P_1 of U_P contained in the exceptional divisor of α_P.

Let $\sigma_P : V_P \to U_P$ be the Kawamata blow up at the singular point P_1. Then, $\mathcal{M}_{V_P} \sim_{\mathbb{Q}} -nK_{V_P}$ by Lemma 0.2.6. Let D be a general surface of the pencil $|-K_{V_P}|$. The proper transform C_{V_P} is the unique base curve of the pencil $|-K_{V_P}|$ and $D \cdot C_{V_P} < 0$. Hence, we have $\mathcal{M} = |-K_X|$ by Theorem 0.2.9 because $\mathcal{M}_{V_P} \sim_{\mathbb{Q}} nD$ by Lemma 0.2.6. But it is impossible by our assumption.

In the exactly same way, we can show that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot consist of the point Q'.

Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point Q. Then, $\mathcal{M}_{U_Q} \sim_{\mathbb{Q}} -nK_{U_Q}$ by Lemma 0.2.6. It follows from Theorem 0.2.4, Lemmas 0.2.7, and 0.2.3 that the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ consists of the singular point O. The divisor $-K_{U_{QO}}$ is nef and big, and hence it follows from Theorem 0.2.4 and Lemma 0.2.7 that the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the singular point O_1 of the variety U_{QO} contained in the exceptional divisor of the β_O.

Let $\sigma_O : V_O \to U_{QO}$ be the Kawamata blow up at the singular point O_1. Then, $\mathcal{M}_{V_O} \sim_{\mathbb{Q}} -nK_{V_O}$ by Lemma 0.2.6. Let H be a general surface of the pencil $|-K_{V_O}|$. The proper transform C_{V_O} is the unique base curve of the pencil $|-K_{V_O}|$ and $H \cdot C_{V_O} = -\frac{1}{24}$, which is impossible by Theorem 0.2.9 because $\mathcal{M} \neq |-K_X|$. $\square$

Consequently, we see that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P, Q\}$. Then, $\mathcal{M}_{U_{PQ}} \sim_{\mathbb{Q}} -nK_{U_{PQ}}$ by Lemma 0.2.6 and it follows from Theorem 0.2.4 and Lemma 0.2.7 that the set $\mathbb{CS}(U_{PQ}, \frac{1}{n}\mathcal{M}_{PQ})$ contains either the singular point Q_2 of the variety U_{PQ} contained in the exceptional divisor of the birational morphism β_Q or the singular point P_2 of the variety U_{PQ} contained in the exceptional divisor of β_P.

Lemma 1.7.4. *The set $\mathbb{CS}(U_{PQ}, \frac{1}{n}\mathcal{M}_{U_{PQ}})$ does not contain the point P_2.*

Proof. Note that the point P_2 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{PQ}.

Suppose that the set $\mathbb{CS}(U_{PQ}, \frac{1}{n}\mathcal{M}_{U_{PQ}})$ contains the point P_2. Let $\pi : V_{PQ} \to U_{PQ}$ be the Kawamata blow up at the point P_2 and D be a general surface of the pencil $|-K_{V_{PQ}}|$.

Then, the proper transform $C_{V_{PQ}}$ is the unique base curve of the pencil $|-K_{V_{PQ}}|$. Because $\mathcal{M}_{V_{PQ}} \sim_{\mathbb{Q}} -nK_W$ by Lemma 0.2.6 and $D \cdot C_{V_{PQ}} = -\frac{1}{24}$, it follows from Theorem 0.2.9 that $\mathcal{M} = |-K_X|$. But it is impossible by our assumption. $\square$

Therefore, it follows from Lemma 0.2.3 that the set $\mathbb{CS}(U_{PQ}, \frac{1}{n}\mathcal{M}_{U_{PQ}})$ consists of the singular point Q_2. In particular, we have $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$, which implies that each surface in the pencil $\mathcal{M}_V$ is contracted to a curve by the elliptic fibration η_0.

Let Q_3 be the singular point of V that is contained in the exceptional divisor of γ_O. It is a quotient singularity of type $\frac{1}{3}(1,1,2)$. Theorem 0.2.4 and Lemma 0.2.7 imply that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point Q_3.

Let $\pi : W \to V$ be the Kawamata blow up at the point Q_3 and D be a general surface in $|-K_W|$. Then, $|-K_W|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_W|$ consists of the irreducible curve C_W. Then, $\mathcal{M} = |-K_X|$ by Theorem 0.2.9 since $\mathcal{M}_W \sim_{\mathbb{Q}} nD$ by Lemma 0.2.6 and $D \cdot C_W < 0$. The obtained contradiction concludes the following:

Proposition 1.7.5. *If $\gimel = 20$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

From now, we consider the case $\gimel = 31$. The threefold X is a hypersurface of degree 16 in $\mathbb{P}(1,1,4,5,6)$ with $-K_X^3 = \frac{2}{15}$. The singularities of X consist of one quotient singularity of type $\frac{1}{2}(1,1,1)$, one singular point P that is a quotient singularity of type $\frac{1}{5}(1,1,4)$, and one singular point Q that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd}
& & V \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[rrrd, "\eta"] & & & \\
& U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & \mathbb{P}(1,1,4), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \\
& X \arrow[rrrruu, dashed, "\psi"] & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,4)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,1,5)$,

- β_Q is the Kawamata blow up with weights $(1,1,5)$ at the point whose image to X is the point Q.
- β_P is the Kawamata blow up with weights $(1,1,4)$ at the point whose image to X is the point P
- β_O is the Kawamata blow up with weights $(1,1,4)$ at the singular point O of U_Q contained in the exceptional divisor of α_Q,
- γ_P is the Kawamata blow up with weights $(1,1,4)$ at the point whose image to X is the point P
- γ_O is the Kawamata blow up with weights $(1,1,4)$ at the singular point of U_{PQ} contained in the exceptional divisor of β_Q,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$t^2w + tf_{11}(x,y,z,w) + f_{16}(x,y,z,w) = 0,$$

where $f_i(x,y,z,w)$ is a general quasihomogeneous polynomial of degree i. Consider the linear system on X defined by the equations

$$\mu w + \sum_{i=0}^{6} \lambda_i x^i y^{6-i} = 0,$$

where $(\mu : \lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5 : \lambda_6) \in \mathbb{P}^7$. It gives us a dominant rational map $\xi : X \dashrightarrow \mathbb{P}(1,1,6)$ defined in the outside of the point P. The normalization of a general fiber is an elliptic curve. Therefore, we have another elliptic fibration as follows:

$$\begin{tikzcd} U_P \arrow[d, "\alpha_P"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta_0"] \\ X \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,1,6), \end{tikzcd}$$

where

- α_P is the Kawamata blow up at the point P with weights $(1,1,4)$,
- β is the Kawamata blow up with weights $(1,1,3)$ at the singular point of the variety U_P that is a quotient singularity of type $\frac{1}{4}(1,1,3)$,
- η_0 is an elliptic fibration.

Proposition 1.7.6. *If $\gimel = 31$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains either the singular point of type $\frac{1}{2}(1,1,1)$ or a curve, we obtain the identity $\mathcal{M} = |-K_X|$ from Lemma 0.3.11 and Corollary 0.3.8. Therefore, due to Lemma 0.3.3, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subseteq \{P, Q\}$.

The proof of Proposition 1.7.6 is similar to that of Proposition 1.7.5. In the case $\gimel = 31$, we do not need Lemmas 1.7.1 and 1.7.2. The only different part is Lemma 1.7.3. However, it works for the point Q in the case $\gimel = 31$ as well. Thus, the following lemma will complete the proof.

Lemma 1.7.7. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P, then $\mathcal{M} = |-K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P. Then, the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ consists of the singular point P_1 of U_P contained in the exceptional divisor of α_P because of Lemmas 0.2.3 and 0.2.7. Furthermore, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must contain the singular point contained in the exceptional divisor of β by Theorem 0.2.4. Consider the Kawamata blow up at this point and apply the same method for Lemma 1.7.3 to get $\mathcal{M} = |-K_X|$. $\square$

1.8. Cases $\gimel = 21$, 35, and 71.

Suppose that $\gimel \in \{21, 35, 71\}$. Then, the threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ always contains the point $O = (0 : 0 : 0 : 1 : 0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, 1, a_4 - a_3)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,a_2),
\end{tikzcd}$$

where

- α is the Kawamata blow up at the point O with weights $(1, 1, a_4 - a_3)$,
- β is the Kawamata blow up with weights $(1, 1, a_4 - a_3 - 1)$ at the point P of U that is a quotient singularity of type $\frac{1}{a_4-a_3}(1, 1, a_4 - a_3 - 1)$,
- η is an elliptic fibration.

By Lemma 0.3.11, if the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point of X different from the singular point O, then the identity $\mathcal{M} = |-K_X|$ holds. Moreover, if it contains a curve, then Corollary 0.3.8 implies $\mathcal{M} = |-K_X|$. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}.$$

due to Theorem 0.2.4 and Lemma 0.3.3

The exceptional divisor $E \cong \mathbb{P}(1, 1, a_4 - a_3)$ of α contains one singular point P that is a quotient singularity of type $\frac{1}{a_4-a_3}(1, 1, a_4 - a_3 - 1)$.

Lemma 1.8.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain a curve.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains a curve Z. Then, Lemma 0.2.7 implies that $Z \in |\mathcal{O}_{\mathbb{P}(1,1,a_4-a_3)}(1)|$, which contradicts Lemma 0.2.3. □

Proposition 1.8.2. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

Proof. It follows from Corollary 0.3.7 and Lemmas 0.3.3, 0.3.11 that we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$. By Lemma 1.8.1, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) = \{P\}$.

The exceptional divisor F of β contains one singular point Q that is a quotient singularity of type $\frac{1}{a_4-2a_3}(1, 1, a_4 - a_3 - 2)$. Because the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty by Theorem 0.2.4, it must contain the point Q.

Let $\pi : Y \to W$ be the Kawamata blow up at the point Q with weights $(1, 1, a_4 - a_3 - 2)$. Easy calculations show that the linear system $|-K_Y|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_Y|$ consists of the irreducible curve C_Y whose image to X is the base curve of the pencil $|-K_X|$. Also, we can easily get

$$-K_Y \cdot C_Y = -K_Y^3 = -\frac{1}{(a_4 - a_3 - 1)(a_4 - a_3 - 2)} < 0.$$

The divisor $(-K_X)^3(-K_Y) + (-K_Y)^3(\alpha \circ \beta \circ \pi)^*(-K_X)$ is nef and big. For a general surface M in $\mathcal{M}_Y$ and a general surface D in $|-K_Y|$,

$$\Big((-K_X)^3(-K_Y) + (-K_Y)^3(\alpha \circ \beta \circ \pi)^*(-K_X)\Big) \cdot D \cdot M = 0,$$

which implies that $\mathcal{M}_Y = |-K_Y|$ by Theorem 0.2.9. Therefore, we obtain the identity $\mathcal{M} = |-K_X|$. □

1.9. Cases $\gimel = 24$ and 46.

We first consider the case $\gimel = 46$. The threefold X is a general hypersurface of degree 21 in $\mathbb{P}(1,1,3,7,10)$ with $-K_X^3 = \frac{1}{10}$. It has only one singular point P that is a quotient singularity of type $\frac{1}{10}(1,3,7)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,1,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,3,7)$,
- β is the Kawamata blow up with weights $(1,3,4)$ at the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,3,4)$,
- γ is the Kawamata blow up with weights $(1,3,1)$ at the singular point of the variety W that is a quotient singularity of type $\frac{1}{4}(1,3,1)$,
- η is an elliptic fibration.

The linear system $|-K_X|$ is a Halphen pencil. Furthermore, we can obtain

Proposition 1.9.1. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. Suppose that we have a Halphen pencil $\mathcal{M}$ different from $|-K_X|$. We are to show that this assumption leads us to a contradiction.

Due to Lemma 0.3.3 and Corollary 0.3.8, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$. Hence, we have $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ by Lemma 0.2.6.

The exceptional divisor $E \cong \mathbb{P}(1,3,7)$ of α contains two singular points P_1 and P_2 of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{7}(1,3,4)$, respectively. The proof of Lemma 1.12.1 shows that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain the point P_1 because $\mathcal{M} \neq |-K_X|$. Hence, Lemma 0.2.7 implies that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the singular point P_2. Therefore, we have $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 0.2.6, which implies that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty.

The exceptional divisor $F \cong \mathbb{P}(1,3,4)$ of β contains two singular points Q_1 and Q_2 of W that are quotient singularities of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{4}(1,3,1)$, respectively. It again follows from Lemma 0.2.7 that either the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1 or the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the point Q_2.

For the convenience, let L be the unique curve contained in $|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)|$ on the surface E and $\bar{L}$ be the unique curve contained in $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$ on the surface F.

Let $\sigma : V_1 \to W$ be the Kawamata blow up at the point Q_1. The pencil $|-K_{V_1}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_{V_1}|$ consists of three irreducible curves C_{V_1}, L_{V_1} and $\bar{L}_{V_1}$.

A general surface D_{V_1} in $|-K_{V_1}|$ is normal. Moreover, the intersection form of the curves C_{V_1}, L_{V_1} and $\bar{L}_{V_1}$ is negatively definite on the surface D_{V_1}. Hence, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ does not contain the point Q_1 by Lemma 0.2.6 and Theorem 0.2.9 because $\mathcal{M}_V \neq |-K_V|$. Thus, we see that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must consist of the point Q_2, which implies that $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ by Lemma 0.2.6. In particular, each member in the pencil $\mathcal{M}_Y$ is contracted to a curve by the elliptic fibration η. Theorem 0.2.4 and Lemma 0.2.7 imply that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point Q of Y contained in the exceptional divisor of the birational morphism γ.

Let $\pi : V_2 \to W$ be the Kawamata blow up at the point Q and D_{V_2} be a general surface in $|-K_{V_2}|$. Then, D_{V_2} is normal, the pencil $|-K_{V_2}|$ is the proper transform of the pencil $|-K_X|$, and the base locus of the pencil $|-K_{V_2}|$ consists of three irreducible curves C_{V_2}, L_{V_2}, and $\bar{L}_{V_2}$. The intersection form of the curves C_{V_2}, L_{V_2}, and $\bar{L}_{V_2}$ is negative-definite on the surface D_{V_2}

because the curves C_{V_2}, L_{V_2}, and $\bar{L}_{V_2}$ are components of a single fiber of the elliptic fibration $\eta\circ\pi|_{D_{V_2}}: D_{V_2}\to\mathbb{P}^1$ that consists of four components. On the other hand, we have

$$\mathcal{M}_{V_2}\Big|_{D_{V_2}}\equiv n(C_{V_2}+L_{V_2}+\bar{L}_{V_2})$$

by Lemma 0.2.6. Thus, we obtain the identity $\mathcal{M}_{V_2}=|-K_{V_2}|$ from Theorem 0.2.9, which is a contradiction to our assumption. □

From now, we consider the case $\gimel=24$. The variety X is a general hypersurface of degree 15 in $\mathbb{P}(1,1,2,5,7)$ with $-K_X^3=\frac{3}{14}$. Its singularities consist of one point that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ and one point P that is a quotient singularity of type $\frac{1}{7}(1,2,5)$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2),
\end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,2,5)$,
- β is the Kawamata blow up with weights $(1,2,3)$ at the singular point of U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$,
- γ is the Kawamata blow up with weights $(1,2,1)$ at the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,2,1)$,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains a curve, then we obtain the identity $\mathcal{M}=|-K_X|$ from Corollary 0.3.8. Therefore, due to Lemmas 0.3.3 and 0.3.11, we may assume that $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})=\{P\}$.

Proposition 1.9.2. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. The proof is the same as that of Proposition 1.9.1. □

1.10. Case $\gimel=25$, hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$.

The threefold X is a general hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$ with $-K_X^3=\frac{5}{28}$. It has two singular points. One is a quotient singularity P of type $\frac{1}{4}(1,1,3)$ and the other is a quotient singularity Q of type $\frac{1}{7}(1,3,4)$.

There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[rrrd, "\eta"] & & & \\
 & U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & \mathbb{P}(1,1,3), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \\
 & X \arrow[rrrruu, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,1,3)$,

- α_Q is the Kawamata blow up at the point Q with weights $(1,3,4)$,
- β_Q is the Kawamata blow up with weights $(1,3,4)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P,
- β_O is the Kawamata blow up with weights $(1,3,1)$ at the singular point O of type $\frac{1}{4}(1,3,1)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,1,3)$ at the point whose image to X is the point P,
- γ_O is the Kawamata blow up with weights $(1,3,1)$ at the singular point of type $\frac{1}{4}(1,3,1)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

Proposition 1.10.1. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

In what follows, we prove Proposition 1.10.1. For the convenience, let D be a general surface in $|-K_X|$.

It follows from [7] that $|-K_X|$ is invariant under the action of the group $\mathrm{Bir}(X)$. Therefore, we may assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical. In fact, we can assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P, Q\Big\}$$

by Lemma 0.3.3 and Corollary 0.3.8.

Lemma 1.10.2. *If the point Q is not contained in $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, then $\mathcal{M} = |-K_X|$.*

Proof. The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ is not empty by Theorem 0.2.4 because $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ by Lemma 0.2.6. Let P_1 be the singular point of the variety U_P contained in the exceptional divisor of the birational morphism α_P. It is a quotient singularity of type $\frac{1}{3}(1,1,2)$. Lemma 0.2.7 implies that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1.

Let $\pi_P : W_P \to U_P$ be the Kawamata blow up at the point P_1 with weights $(1,1,2)$. We can easily check that $|-K_{W_P}|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_W|$ consists of the irreducible curve C_{W_P}. We can also see $D_{W_P} \cdot C_{W_P} = -K_{W_P}^3 = -\frac{1}{14} < 0$. Hence, Theorem 0.2.9 implies the identity $\mathcal{M} = |-K_X|$ because $\mathcal{M}_{W_P} \sim_{\mathbb{Q}} nD_{W_P}$ by Lemma 0.2.6. ∎

The exceptional divisor $E \cong \mathbb{P}(1,3,4)$ of the birational morphism α_Q contains two singular points O and Q_1 that are quotient singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$. Let L be the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$ on the surface E.

Due to Lemma 1.10.2, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point Q. The proof of Lemma 1.10.2 also shows that the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ cannot consist of the single point $\bar{P}$ whose image to X is the point P. It implies

$$\mathbb{CS}\Big(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}\Big) \cap \Big\{O, Q_1\Big\} \neq \varnothing$$

by Theorem 0.2.4 and Lemma 0.2.7.

Lemma 1.10.3. *If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains both the point O and the point Q_1, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\gamma_Q : W_Q \to U_{QO}$ be the Kawamata blow up with weights $(1,2,1)$ at the point whose image to U_Q is the point Q_1.

The proper transform D_{W_Q} is irreducible and normal. The base locus of the pencil $|-K_{W_Q}|$ consists of the irreducible curves C_{W_Q} and L_{W_Q}. On the other hand, we have

$$\mathcal{M}_{W_Q}\Big|_{D_{W_Q}} \equiv -nK_{W_Q}\Big|_{D_{W_Q}} \equiv nC_{W_Q} + nL_{W_Q},$$

but the intersection form of the curves L_{W_Q} and C_{W_Q} on the normal surface D_{W_Q} is negative-definite. Then, Theorem 0.2.9 completes the proof. □

It follows from Lemma 0.2.7 that we may assume the following possibilities:

- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, O\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{O\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$.

The exceptional divisor $F \cong \mathbb{P}(1,3,1)$ of β_O contains one singular point Q_2 that is a quotient singularity of type $\frac{1}{3}(1,2,1)$.

Lemma 1.10.4. *If $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{O\}$, then $\mathcal{M} = |-K_X|$.*

Proof. The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the singular point Q_2 by Theorem 0.2.4 and Lemma 0.2.7.

Let $\gamma : W \to U_{QO}$ be the Kawamata blow up at the point Q_2 with weights $(1,2,1)$. Then, $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 0.2.6 and the base locus of the pencil $|-K_W|$ consists of the curves C_W and L_W. The proper transform D_W is irreducible and normal, the equivalence $\mathcal{M}_W\Big|_{D_W} \equiv nC_W + nL_W$ holds, but the equalities

$$C_W^2 = -\frac{7}{12},\quad L_W^2 = -\frac{5}{6},\quad C_W \cdot L_W = \frac{2}{3}$$

hold on the surface D_W. So, the intersection form of the curves C_W and L_W on the normal surface D_W is negative-definite, which implies $\mathcal{M} = |-K_X|$ by Theorem 0.2.9. □

Lemma 1.10.5. *If $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, O\}$, then $\mathcal{M} = |-K_X|$.*

Proof. We have $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, which implies that every surface of the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η. In particular, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain curves because the exceptional divisors of $\beta_O \circ \gamma_P$ are sections of η.

Because of Theorem 0.2.4 and Lemmas 0.2.7, 1.10.4, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point P_2 of Y contained in the exceptional divisor γ_P. Let $\sigma_P : Y_P \to Y$ be the Kawamata blow up at the point P_2 with weights $(1,2,1)$. Then, $\mathcal{M}_{Y_P} \sim_{\mathbb{Q}} -nK_{Y_P}$ but the base locus of the pencil $|-K_{Y_P}|$ consists of the irreducible curves C_{Y_P} and L_{Y_P}.

The proper transform D_{Y_P} is normal and $\mathcal{M}_{Y_P}|_{D_{Y_P}} \equiv nC_{Y_P} + nL_{Y_P}$. The intersection form of the curves C_{Y_P} and L_{Y_P} on the normal surface D_{Y_P} is negative-definite because the curves are contained in a fiber of $\eta \circ \sigma_P|_{D_{Y_P}}$ that consists of three irreducible components. Therefore, we obtain the identity $\mathcal{M} = |-K_X|$ from Theorem 0.2.9. □

Thus, to conclude the proof of Proposition 1.12.2, we may assume the following possibilities:

- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$;
- $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$.

The hypersurface X can be given by the equation

$$w^2y + wt^2 + wtf_4(x,y,z) + wf_8(x,y,z) + tf_{11}(x,y,z) + f_{15}(x,y,z) = 0,$$

where f_i is a general quasihomogeneous polynomial of degree i.

Lemma 1.10.6. *The case $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$ never happens.*

Proof. Suppose that $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{Q_1\}$. Let $\pi : V \to U_Q$ be the Kawamata blow up at the point Q_1 with weights $(1,2,1)$.

Let G be the exceptional divisor of the birational morphism π. The proof of Lemma 1.10.4 implies that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ does not contain the singular point of V contained in the exceptional divisor G. So, the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal by Lemma 0.2.7 and Corollary 0.3.4.

We have

$$\begin{cases} (\alpha_Q\circ\pi)^*\big(-K_X\big)\sim_{\mathbb{Q}} S_V+\dfrac{3}{7}G+\dfrac{1}{7}E_V,\\ (\alpha_Q\circ\pi)^*\big(-K_X\big)\sim_{\mathbb{Q}} S_V^y+\dfrac{10}{7}G+\dfrac{8}{7}E_V,\\ (\alpha_Q\circ\pi)^*\big(-3K_X\big)\sim_{\mathbb{Q}} S_V^z+\dfrac{2}{7}G+\dfrac{3}{7}E_V,\\ (\alpha_Q\circ\pi)^*\big(-4K_X\big)\sim_{\mathbb{Q}} S_V^t+\dfrac{5}{7}G+\dfrac{4}{7}E_V,\\ (\alpha_Q\circ\pi)^*\big(-7K_X\big)\sim_{\mathbb{Q}} S_V^w. \end{cases}$$

The equivalences imply

$$\begin{cases} (\alpha_Q\circ\pi)^*\Big(\dfrac{y}{x}\Big)\in|S_V|,\\ (\alpha_Q\circ\pi)^*\Big(\dfrac{yz}{x^4}\Big)\in|4S_V|,\\ (\alpha_Q\circ\pi)^*\Big(\dfrac{yt}{x^5}\Big)\in|5S_V|,\\ (\alpha_Q\circ\pi)^*\Big(\dfrac{y^3w}{x^{10}}\Big)\in|10S_V|, \end{cases}$$

and hence the complete linear system $|-20K_V|$ induces a birational map $\chi_1: V \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,4,5,10)$, which implies that the divisor $-K_V$ is big.

The base locus of the pencil $|-K_V|$ consists of the irreducible curves C_V and L_V. It follows from [21] that there is a composition of antiflips $\zeta: V \dashrightarrow V'$ such that ζ is regular in the outside of $C_V\cup L_V$ and the anticanonical divisor $-K_{V'}$ is nef and big. The singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ are terminal because the rational map ζ is a log flop with respect to the log pair $(V, \frac{1}{n}\mathcal{M}_V)$, which contradicts Theorem 0.2.4. □

Now, we suppose $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{\bar{P}, Q_1\}$. Let $\sigma: U\to U_{PQ}$ be the Kawamata blow up with weights $(1,2,1)$ at the point $\bar{Q}_1$ whose image to U_Q is the point Q_1. Let $\bar{E}$ and $\tilde{E}$ be the exceptional divisors of α_P and σ, respectively. Then,

$$\begin{cases} S_U\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-K_X\big)-\dfrac{3}{7}\tilde{E}-\dfrac{1}{7}E_U-\dfrac{1}{4}\bar{E}_U\sim_{\mathbb{Q}}-K_W,\\ S_U^y\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-K_X\big)-\dfrac{10}{7}\tilde{E}-\dfrac{8}{7}E_U-\dfrac{1}{4}\bar{E}_U,\\ S_U^z\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-3K_X\big)-\dfrac{2}{7}\tilde{E}-\dfrac{3}{7}E_U-\dfrac{3}{4}\bar{E}_U,\\ S_U^t\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-4K_X\big)-\dfrac{5}{7}\tilde{E}-\dfrac{4}{7}E_U,\\ S_U^w\sim_{\mathbb{Q}}(\alpha_P\circ\beta_Q\circ\sigma)^*\big(-7K_X\big)-\dfrac{11}{4}\bar{E}_U. \end{cases}$$

The equivalences imply that the pull-backs of rational functions

$$\frac{y}{x},\ \frac{yz}{x},\ \frac{y^3w}{x^{10}},\ \frac{y^4tw}{x^{15}}$$

are contained in the linear system $|aS_U|$, where $a = 1$, 4, 10, and 15, respectively. Therefore, the linear system $|-60K_U|$ induces a birational map $\chi_2: U \dashrightarrow X''$ such that the variety X'' is a hypersurface of degree 30 in $\mathbb{P}(1,1,4,10,15)$, which implies that the anticanonical divisor $-K_U$ is big. However, the proof of Lemma 1.10.4 shows that the singularities of $(U, \frac{1}{n}\mathcal{M}_U)$ are terminal. Then, we can obtain a contradiction in the same way as in the proof of Lemma 1.10.6.

1.11. Case $\daleth = 26$, hypersurface of degree 15 in $\mathbb{P}(1,1,3,5,6)$.

The threefold X is a general hypersurface of degree 15 in $\mathbb{P}(1,1,3,5,6)$ with $-K_X^3 = \frac{1}{6}$. The singularities of the hypersurface X consist of two points P_1 and P_2 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and one point Q that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

For each of the singular points P_1 and P_2, we have a commutative diagram

$$\begin{tikzcd} & U_i \arrow[dl, "\sigma_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\xi_i"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where

- ξ_i is a projection,
- σ_i is the Kawamata blow up at the point P_i with weights $(1,1,2)$,
- η_i is an elliptic fibration.

Note that the threefold X has a unique birational automorphism which is not biregular. It is a quadratic involution that is defined in the outside of the point Q. Moreover, it interchanges the points P_1 and P_2. The linear system that induces the rational map ξ_1 is transformed into the linear system that induces the rational map ξ_2 by the quadratic involution and vice versa.

We have another elliptic fibration

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,3) \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,5)$,
- β is the Kawamata blow up with weights $(1,1,4)$ at the singular point Q_1 of the variety U contained in the exceptional divisor of α,
- γ is the Kawamata blow up with weights $(1,1,3)$ at the singular point Q_2 contained in the exceptional divisor of the birational morphism β,
- η is an elliptic fibration.

It follows from [7] that the pencil $|-K_X|$ is invariant under the action of the group $\mathrm{Bir}(X)$. Hence, we may assume that the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical. In fact, we can assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P_1, P_2, Q\Big\},$$

due to Lemma 0.3.3 and Corollary 0.3.8.

Lemma 1.11.1. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains either the point P_1 or the point P_2, then $\mathcal{M} = |-K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P_1. Because $\mathcal{M}_{U_1} \sim_{\mathbb{Q}} -nK_{U_1}$ by Lemma 0.2.6, each surface in the pencil $\mathcal{M}_{U_1}$ is contracted to a curve by the morphism η_1 and hence the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ does not contain curves.

Let $\bar{P}_2$ and $\bar{Q}$ be the points on U_1 whose images to X are the points P_2 and Q, respectively, and O be the singular point of U_1 contained in the exceptional divisor of σ_1. Then,

$$\mathbb{CS}\Big(U_1, \frac{1}{n}\mathcal{B}\Big) \cap \Big\{\bar{P}_2, \bar{Q}, O\Big\} \neq \varnothing$$

by Theorem 0.2.4 and Lemma 0.2.7. We consider only the case when the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ contains the point O because the other cases are similar. Suppose that the set $\mathbb{CS}(U_1, \frac{1}{n}\mathcal{M}_{U_1})$ contains the point O.

Let $\pi : V \to U_1$ be the Kawamata blow up at the point O with weights $(1,1,1)$. Then, the base locus of the pencil $|-K_V|$ consists of the curve C_V. Let D_V be a general surface in $|-K_V|$. Then, D_V is normal and $C_V^2 = -\frac{1}{2}$ on the surface D_V, which implies that $\mathcal{M} = |-K_X|$ by Lemma 0.2.6 and Theorem 0.2.9. $\square$

Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point Q.

The exceptional divisor E of the birational morphism α contains a singular point Q_1 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$. The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 0.2.4.

Lemma 1.11.2. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q_1.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains a subvariety $Z \subset U$ that is different from the singular point Q_1. Then, the subvariety Z is a curve with $-K_U \cdot Z = \frac{1}{5}$ by Lemma 0.2.7, which is impossible by Lemma 0.2.3 because $-K_U^3 = \frac{2}{15}$. $\square$

Let G be the exceptional divisor of the birational morphism γ and Q_3 be the singular point of G. Then, Theorem 0.2.4, Lemmas 0.2.6, and 0.2.7 imply that $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$. Each member in the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η and $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) \neq \varnothing$.

Lemma 1.11.3. *The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{H})$ consists of the point Q_3.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains a subvariety Z of the variety Y that is different from the point Q_3. Let F be the exceptional divisor of the birational morphism β. The base locus of the pencil $\mathcal{M}_Y$ does not contain curves contained in F since F is a section of η. Then, Z is an irreducible curve such that the curve $\gamma(Z)$ is a ruling of the cone F by Lemma 0.2.7. Thus, we have $-K_W \cdot \gamma(C) = \frac{1}{4}$, which is impossible by Lemma 0.2.3. $\square$

Let $\gamma_1 : Y_1 \to Y$ be the Kawamata blow up at the point Q_3 with weights $(1,1,2)$. Then, the base locus of the pencil $|-K_{Y_1}|$ consists of the curve C_{Y_1}. Let D_{Y_1} be a general surface in $|-K_{Y_1}|$. Then, D_{Y_1} is normal and $C_{Y_1}^2 = -\frac{1}{6}$ on the surface D_{Y_1}, which implies that $\mathcal{M} = |-K_X|$ by Lemma 0.2.6 and Theorem 0.2.9.

Therefore, we have obtained

Proposition 1.11.4. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

1.12. Cases $\gimel = 29$, 50, and 67.

Suppose that $\gimel \in \{29, 50, 67\}$. Then, the threefold $X \subset \mathbb{P}(1,1,a_2,a_3,a_4)$ always contains the point $O = (0:0:0:1:0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, a_2, a_3 - a_2)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,a_2), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1, a_2, a_3 - a_2)$,
- β is the Kawamata blow up with weights $(1, a_2, 1)$ at the point P of U that is a quotient singularity of type $\frac{1}{a_3 - a_2}(1, a_2, 1)$,
- η is an elliptic fibration.

The exceptional divisor E of the birational morphism α contains two singular points P and Q that are quotient singularity of types $\frac{1}{a_3-a_2}(1,a_2,1)$ and $\frac{1}{a_2}(1,a_2-1,1)$, respectively. The base locus of $|-K_X|$ consists of the irreducible curve C defined by $x=y=0$. The base curve of $|-K_U|$ consists of the proper transform C_U and the unique irreducible curve L in $|\mathcal{O}_{\mathbb{P}(1,a_2,a_3-a_2)}(1)|$ on the surface E.

Lemma 1.12.1. *If the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains the point Q, then $\mathcal{M}=|-K_X|$.*

Proof. Let $\pi: Y\to U$ be the Kawamata blow up at the point Q with weights $(1,a_2-1,1)$ and G_Q be its exceptional divisor. Then, the base locus of the pencil $|-K_Y|$ consists of the irreducible curves C_Y and L_Y. Let D be a general surface in $|-K_X|$. We then have

$$\begin{cases} D_Y\sim_{\mathbb{Q}}(\alpha\circ\pi)^*\Big(-K_X\Big)-\dfrac{1}{a_3}\pi^*(E)-\dfrac{1}{a_2}G_Q,\\ S_Y\sim_{\mathbb{Q}}(\alpha\circ\pi)^*\Big(-K_X\Big)-\dfrac{1}{a_3}\pi^*(E)-\dfrac{1}{a_2}G_Q,\\ E_Y\sim_{\mathbb{Q}}\pi^*(E)-\dfrac{1}{a_2}G_Q\end{cases}$$

and we also have $S_Y\cdot D_Y=C_Y+L_Y$ and $E_Y\cdot D_Y=L_Y$. It then follows that

$$\begin{cases} D_Y\cdot L_Y=E_Y\cdot D_Y^2=E_Y\cdot K_Y^2=-\dfrac{a_3+1}{a_2(a_3-a_2)(a_2-1)}<0,\\ D_Y\cdot C_Y=(-K_Y)^3-D_Y\cdot L_Y\\ \qquad=\dfrac{a_4}{a_2a_3}-\dfrac{1}{a_2a_3(a_3-a_2)}-\dfrac{1}{a_2(a_2-1)}+\dfrac{a_3+1}{a_2(a_3-a_2)(a_2-1)}\leq 0.\end{cases}$$

The divisors $-K_U$ and $-K_X$ are nef and big. Moreover,

$$\begin{cases} -K_U\cdot L=E\cdot K_U^2=E\cdot K_U^2=\dfrac{1}{a_2(a_3-a_2)},\\ -K_U\cdot C_U=S_U\cdot D_U^2+K_U\cdot L=-K_U^3+K_U\cdot L=\dfrac{1}{a_2a_3}\Big(a_4-\dfrac{1+a_3}{a_3-a_2}\Big),\\ -K_X\cdot C=-K_X^3=\dfrac{a_4}{a_2a_3}.\end{cases}$$

One can easily check that

$$\lambda:=\frac{D_Y\cdot L_Y}{K_U\cdot L}>0,\quad \mu:=\frac{(D_Y\cdot C_Y)(K_U\cdot L)-(D_Y\cdot L_Y)(K_U\cdot C_U)}{(K_X\cdot C)(K_U\cdot L)}\geq 0.$$

Then, $B=\pi^*(-\lambda K_U)+(\alpha\circ\pi)^*(-\mu K_X)+D_Y$ is a nef and big divisor with $B\cdot L_Y=B\cdot C_Y=0$. Therefore, $B\cdot D_Y\cdot M_Y=0$, where M_Y is a general surface in $\mathcal{M}_Y$, and hence $\mathcal{M}=|-K_X|$ by Theorem 0.2.9. □

The exceptional divisor F of the birational morphism β contains one singular point P_1 of the threefold W that is a quotient singularity of type $\frac{1}{a_2}(1,a_2-1,1)$.

Proposition 1.12.2. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. By Lemma 0.3.11, if the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains a singular point of X different from the singular point O, then the identity $\mathcal{M}=|-K_X|$ holds. Moreover, if it contains a curve, then Corollary 0.3.8 implies $\mathcal{M}=|-K_X|$. Therefore, we may assume that

$$\mathbb{CS}\Big(X,\frac{1}{n}\mathcal{M}\Big)=\Big\{O\Big\}.$$

due to Theorem 0.2.4 and Lemma 0.3.3.

Furthermore, Lemma 0.2.7 implies that either $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)=\{P\}$ or $Q\in\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$. The latter case implies $\mathcal{M}=|-K_X|$ by Lemma 1.12.1. Suppose that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ consists of the point P. Then, the set $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)$ contains the point P_1. Let $\sigma: V\to W$ be the

Kawamata blow up at the point P_1. Then, the base locus of the pencil $|-K_V|$ consists of the irreducible curves C_V and L_V. Applying the same method as in Lemma 1.12.1, we obtain the identity $\mathcal{M} = |-K_X|$. $\square$

1.13. Cases $\gimel = 34,\ 53,\ 70,$ and 88.

Suppose $\gimel \in \{34, 53, 70, 88\}$. Then, the hypersurface X has a singular point P of type $\frac{1}{a_2+1}(1, 1, a_2)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,a_2)
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point P with weights $(1, 1, a_2)$,
- η is an elliptic fibration.

We may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P by Lemmas 0.3.3, 0.3.11 and Corollary 0.3.8.

Proposition 1.13.1. *If $\gimel \in \{34, 53, 70, 88\}$, then the linear system $|-K_X|$ a unique Halphen pencil.*

Proof. The singularities of the log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ are not terminal by Theorem 0.2.4. The base locus of the pencil $\mathcal{M}_Y$ does not contain curves that are not contained in a fiber of the elliptic fibration η. Hence, it follows from Lemma 0.2.7 that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point Q contained in the exceptional divisor of π, which is a quotient singularity of type $\frac{1}{a_2}(1, 1, a_2 - 1)$ on Y.

Let $\alpha : U \to Y$ be the Kawamata blow up at the point P with weights $(1, 1, a_2 - 1)$. Then, the pencil $|-K_U|$ is the proper transform of the pencil $|-K_X|$. Its base locus consists of the irreducible curve C_U. Moreover, $-K_U \cdot C_U = -K_U^3 < 0$ and $\mathcal{M}_U \equiv -nK_U$. We then obtain $\mathcal{M} = |-K_X|$ from Theorem 0.2.9. $\square$

1.14. Case $\gimel = 36$, hypersurface of degree 18 in $\mathbb{P}(1, 1, 4, 6, 7)$.

Let X be the hypersurface given by a general quasihomogeneous equation of degree 18 in $\mathbb{P}(1, 1, 4, 6, 7)$ with $-K_X^3 = \frac{3}{28}$. Then, the singularities of X consist of two singular points P and Q that are quotient singularities of types $\frac{1}{7}(1, 1, 6)$ and $\frac{1}{4}(1, 1, 3)$, respectively, and one point of type $\frac{1}{2}(1, 1, 1)$.

There is a commutative diagram

$$\begin{tikzcd}
U_P \arrow[d, "\alpha_P"'] & V \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4)
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the weighted blow up at the point P with weights $(1, 1, 6)$,
- β is the Kawamata blow up with weights $(1, 1, 5)$ at the singular point O_1 of the variety U_P contained in the exceptional divisor of the birational morphism α_P,
- γ is the Kawamata blow up with weights $(1, 1, 4)$ at the singular point O_2 of the variety V contained in the exceptional divisor of the birational morphism β,

- η is an elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation of degree 18

$$z^3t + z^2f_{10}(x,y,t,w) + zf_{14}(x,y,t,w) + f_{18}(x,y,t,w) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\xi : X \dashrightarrow \mathbb{P}^7$ be the rational map given by the linear system of divisors cut on the hypersurface X by the equations

$$\mu t + \sum_{i=0}^{6} \lambda_i x^i y^{6-i} = 0$$

, where $(\mu : \lambda_0 : \cdots : \lambda_6) \in \mathbb{P}^7$. Then, we obtain another commutative diagram

$$\begin{tikzcd}
 & W \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] \arrow[drr, "\eta_1"] & & \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & \mathbb{P}(1,1,6), \\
 & X \arrow[urr, dashed, "\xi"'] & &
\end{tikzcd}$$

where

- α_P is the Kawamata blow up at the singular point P with weights $(1,1,6)$,
- α_Q is the Kawamata blow up at the singular point Q with weights $(1,1,3)$,
- β_P is the Kawamata blow up with weights $(1,1,6)$ at the point P_1 whose image to X is the point P,
- β_Q is the Kawamata blow up with weights $(1,1,3)$ at the point Q_1 whose image to X is the point Q,
- η_1 is an elliptic fibration.

If it contains either the singular point of type $\frac{1}{2}(1,1,1)$ or a curve, then Lemma 0.3.11 and Corollary 0.3.8 imply $\mathcal{M} = |-K_X|$. Therefore, we may also assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}.$$

The exceptional divisor E_Q of the birational morphism α_Q contains a unique singular point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.

Lemma 1.14.1. *If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains the point O, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\beta_O : U_O \to U_Q$ be the Kawamata blow up at the point O with weights $(1,1,2)$. Then, the proper transform $\mathcal{D}_{U_O}$ of the pencil $|-K_X|$ by the birational morphism $\alpha_Q \circ \beta_O$ has a unique base curve C_{U_O}. Because a general surface in $\mathcal{D}_{U_O}$ is normal and the self-intersection of C_{U_O} on a general surface in $\mathcal{D}_{U_O}$ is negative, we obtain $\mathcal{M} = |-K_X|$. □

Lemma 1.14.2. *If $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q}) = \{P_1\}$, then $\mathcal{M} = |-K_X|$.*

Proof. The proper transform $\mathcal{M}_W$ must contain the singular point in the exceptional divisor F_P of the birational morphism β_P that is a quotient singularity of type $\frac{1}{6}(1,1,5)$. Let $\sigma_1 : W_1 \to W$ be the Kawamata blow up at the singular point with weights $(1,1,5)$. Then, the proper transform $\mathcal{D}_{W_1}$ of the pencil $|-K_X|$ has a unique base curve C_{W_1}. Because a general surface in $\mathcal{D}_{W_1}$ is normal and the self-intersection of C_{W_1} on a general surface in $\mathcal{D}_{W_1}$ is negative, we obtain $\mathcal{M} = |-K_X|$. □

Meanwhile, the exceptional divisor E_P of the birational morphism α_P contains one singular point O_1 of type $\frac{1}{6}(1,1,5)$.

Lemma 1.14.3. *The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ cannot contain a curve.*

Proof. Because $-K_{U_P}^3 = \frac{1}{12}$ and $E_P \cdot K_{U_P}^2 = \frac{1}{6}$, the statement immediately follows from Lemma 0.2.3. □

Therefore, we may assume that

$$\mathbb{CS}\Big(U_1, \frac{1}{n}\mathcal{M}_{U_1}\Big) = \Big\{O_1\Big\}.$$

By the same method of Lemma 1.14.3, we may assume that $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V) = \{O_2\}$, and hence the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must contain the singular point contained in the exceptional divisor of γ that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. By considering Kawamata blow up at this point with weights $(1,1,3)$ and the proper transform of the pencil $|-K_X|$, one can easily conclude the following:

Proposition 1.14.4. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

1.15. Cases $\daleth = 47$, 54, and 62.

We first consider the case $\daleth = 47$ which is more complicated than the other. The threefold X is a general hypersurface of degree 21 in $\mathbb{P}(1,1,5,7,8)$ with $-K_X^3 = \frac{3}{40}$. The singularities of X consist of one point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and one point Q that is a quotient singularity of type $\frac{1}{8}(1,1,7)$.

We have the following commutative diagram:

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rrrr, dashed, "\psi"'] & & & & \mathbb{P}(1,1,5),
\end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,7)$,
- β is the Kawamata blow up with weights $(1,1,6)$ at the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,1,6)$,
- γ is the Kawamata blow up with weights $(1,1,5)$ at the singular point of W that is a quotient singularity of type $\frac{1}{6}(1,1,5)$,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2z + \sum_{i=0}^{2} wz^i g_{13-5i}(x,y,t) + \sum_{i=0}^{3} z^i g_{21-5i}(x,y,t) = 0,$$

where $g_i(x,y,t)$ is a quasihomogeneous polynomial of degree i. The base locus of the pencil $|-K_X|$ consists of the irreducible curve C cut out on X by the equations $x = y = 0$. Let D be a general surface in $|-K_X|$. Then, D is smooth at a generic point of C. Note that

$$\mathbb{CS}\Big(X, |-K_X|\Big) = \Big\{P, Q, C\Big\}.$$

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then the identity $\mathcal{M} = |-K_X|$ holds due to Corollary 0.3.8. Furthermore, by Lemma 0.3.3, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P, Q\Big\}.$$

Lemma 1.15.1. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point Q.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain Q. Then, $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$, and hence $\mathcal{M} \neq |-K_X|$.

Let $\alpha_P : U_P \to X$ be the Kawamata blow up at the point P. The exceptional divisor of $E_P \cong \mathbb{P}(1, 2, 3)$ contains two singular points P_1 and P_2 of the threefold U_P that are quotient singularities of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 2, 1)$, respectively. For the convenience, let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$ on the surface E_P.

If the log pair $(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ is not terminal, then it is not terminal either at the point P_1 or at the point P_2. In such cases, the proof of Lemma 1.12.1 leads us to the identity $\mathcal{M} = |-K_X|$, which contradicts our assumption. Therefore, the log pair must be terminal.

The base locus of the pencil $|-K_{U_P}|$ consists of the curve C_{U_P} and the irreducible curve L. The inequalities $-K_{U_P} \cdot C_{U_P} < 0$ and $-K_{U_P} \cdot L > 0$ implies that the curve C_{U_P} is the only curve on the variety U_P that has negative intersection with the divisor $-K_{U_P}$. It follows from [21] that the antiflip $\zeta_1 : U_P \dashrightarrow U'_P$ along the curve C_{U_P} exists. The divisor $-K_{U'_P}$ is nef. The log pair $(U'_P, \frac{1}{n}\mathcal{M}_{U'_P})$ is terminal because the log pair $(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ is so. On the other hand, the pull-backs of the rational functions 1, $\frac{y}{x}$, $\frac{zy}{x^6}$, $\frac{ty}{x^8}$ and $\frac{yw}{x^9}$ induce a birational map χ_1 of U_P onto a hypersurface X' of degree 24 in $\mathbb{P}(1, 1, 6, 8, 9)$, which implies that $-K_{U_P}$ is big. Hence, the divisor $-K_{U'_P}$ is big as well, which contradicts Theorem 0.2.4. □

The exceptional divisor E of the birational morphism α contains the singular point Q_1 of the threefold U that is a quotient singularity of type $\frac{1}{7}(1, 1, 6)$. Lemmas 0.2.3 and 0.2.7 show that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain any other subvariety of the surface E than the point Q_1.

The pencil $|-K_U|$ is the proper transform of the pencil $|-K_X|$ and the base locus of $|-K_U|$ consists of the curve C_U.

Lemma 1.15.2. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q_1.*

Proof. Suppose that it does not contain the point Q_1. Then, it follows from Theorem 0.2.4 and Lemma 0.2.7 that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point $\bar{P}$ whose image to X is the point P because the divisor $-K_U$ is nef and big.

Let $\beta_P : W_P \to U$ be the Kawamata blow up at the point $\bar{P}$ with weights $(1, 2, 3)$ and $F_P \cong \mathbb{P}(1, 2, 3)$ be its exceptional divisor. The proof of Lemma 1.15.1 implies that the log pair $(W_P, \frac{1}{n}\mathcal{M}_{W_P})$ is terminal.

The base locus of the pencil $|-K_{W_P}|$ consists of the irreducible curve C_{W_P} and the unique irreducible curve $\bar{L}$ of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$ on the surface F_P. Hence, there is an antiflip $\zeta_2 : W_P \dashrightarrow W'_P$ along the curve C_{W_P}, the divisor $-K_{W'_P}$ is nef, and the log pair $(W'_P, \frac{1}{n}\mathcal{M}_{W'_P})$ is terminal.

The pull-backs of the rational functions 1, $\frac{y}{x}$, $\frac{zy}{x^6}$, $\frac{ty}{x^8}$ and $\frac{wzy^2}{x^{15}}$ induce a birational map $\chi_2 : W_P \dashrightarrow X''$ such that X'' is a hypersurface of degree 30 in $\mathbb{P}(1, 1, 6, 8, 15)$. We have

$$\begin{cases} S_{W_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-K_X) - \dfrac{1}{8}E_{W_P} - \dfrac{1}{5}F_P, \\ S^y_{W_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-K_X) - \dfrac{1}{8}E_{W_P} - \dfrac{6}{5}F_P, \\ S^z_{W_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-5K_X) - \dfrac{13}{8}E_{W_P}, \\ S^t_{W_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-7K_X) - \dfrac{7}{8}E_{W_P} - \dfrac{2}{5}F_P, \\ S^w_{W_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-8K_X) - \dfrac{3}{5}F_P, \end{cases}$$

which implies that X'' is the anticanonical model of W'_P. In particular, the divisor $-K_{W'_P}$ is big, which is impossible by Theorem 0.2.4. □

The exceptional divisor F of the birational morphism β contains a singular point Q_2 of W that is a quotient singularity of type $\frac{1}{6}(1,1,5)$. The divisor $-K_W$ is nef and big, and hence the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty by Theorem 0.2.4.

Lemma 1.15.3. *The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2.*

Proof. Suppose that it does not contain the point Q_2. Then, it follows from Theorem 0.2.4, Lemmas 0.2.3, and 0.2.7 that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the point $\tilde{P}$ whose image to X is the point P.

Let $\pi : Y_P \to W$ be the Kawamata blow up at the point $\tilde{P}$ and G_P be its exceptional divisor. Then, the proof of Lemma 1.15.1 shows that the log pair $(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ is terminal.

We have

$$\begin{cases} S_{Y_P} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \dfrac{1}{8}E_{Y_P} - \dfrac{1}{4}F_{Y_P} - \dfrac{1}{5}G_P, \\ S^y_{Y_P} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-K_X) - \dfrac{1}{8}E_{Y_P} - \dfrac{1}{4}F_{Y_P} - \dfrac{6}{5}G_P, \\ S^z_{Y_P} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-5K_X) - \dfrac{13}{8}E_{Y_P} - \dfrac{9}{4}F_{Y_P}, \\ S^t_{Y_P} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-7K_X) - \dfrac{7}{8}E_{Y_P} - \dfrac{3}{4}F_{Y_P} - \dfrac{2}{5}G_P, \\ S^w_{Y_P} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\pi)^*(-8K_X) - \dfrac{3}{5}G_P, \end{cases}$$

which implies that the anticanonical model of Y_P is a hypersurface of degree 42 in $\mathbb{P}(1,1,6,14,21)$ with canonical singularities because the pull-backs of the rational functions $\frac{y}{x}$, $\frac{zy}{x^6}$, $\frac{tzy^2}{x^{14}}$ and $\frac{wz^2y^3}{x^{21}}$ are contained in the linear systems $|S_{Y_P}|$, $|6S_{Y_P}|$, $|14S_{Y_P}|$ and $|21S_{Y_P}|$, respectively. In particular, the divisor $-K_{Y_P}$ is big.

The base locus of the linear system $|-42K_{Y_P}|$ consists of the curve C_{Y_P} and $-K_{Y_P}\cdot C_{Y_P} < 0$. Therefore, the existence of the antiflip $\zeta_3 : Y_P \dashrightarrow Y'_P$ along the curve C_{Y_P} follows from [21]. The divisor $-K_{Y'_P}$ is nef and big. However, this contradicts Theorem 0.2.4 because the log pair $(Y'_P, \frac{1}{n}\mathcal{M}_{Y'_P})$ is terminal. □

The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ is not empty. If it contains the singular point Q_3 contained in the exceptional divisor of γ, then it easily follows from Theorem 0.2.9 that $\mathcal{M} = |-K_X|$. If the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain the point Q_3, then Lemmas 0.2.3 and 0.2.7 imply that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ consists of the point $\hat{P}$ whose image to X is the point P.

Let $\sigma : V \to Y$ be the Kawamata blow up at the point $\hat{P}$. Then, the proof of Lemma 1.15.1 shows that the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal. The proof of Lemma 1.15.3 implies that the anticanonical model of V is the surface $\mathbb{P}(1,1,6)$. On the other hand, there is an antiflip $\zeta : V \dashrightarrow V'$ along the curve C_V, which implies that the linear system $|-rK_{V'}|$ induces a surjective morphism $V' \to \mathbb{P}(1,1,6)$. However, it contradicts Theorem 0.2.4 because the singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ are terminal.

Consequently, we have shown

Proposition 1.15.4. *If $\daleth = 47$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

Next, we consider the case $\daleth = 54$. The variety X is a general hypersurface of degree 24 in $\mathbb{P}(1,1,6,8,9)$ with $-K_X^3 = \frac{1}{18}$. The singularities of X consist of one point that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, one point that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and a point Q that is a quotient singularity of type $\frac{1}{9}(1,1,8)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,8)$,
- β is the Kawamata blow up with weights $(1,1,7)$ at the singular point of U that is a quotient singularity of type $\frac{1}{8}(1,1,7)$,
- γ is the Kawamata blow up with weights $(1,1,6)$ at the singular point of W that is a quotient singularity of type $\frac{1}{7}(1,1,6)$,
- η is and elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains either one of the singular points of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ or a curve, then we obtain $\mathcal{M} = |-K_X|$ from Lemma 0.3.11 and Corollary 0.3.8. Therefore, due to Lemma 0.3.3, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{Q\Big\}.$$

Proposition 1.15.5. *If $\gimel = 54$, then the linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. Unlike the case $\gimel = 47$, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain any other point than the point Q, which makes our situation far simpler than the case $\gimel = 47$. In this case, Lemmas 1.15.1, 1.15.2, 1.15.3 are automatically satisfied, so that the proof of Proposition 1.15.4 proves this statement. □

From now, we consider the case $\gimel = 62$. The variety X is a general hypersurface of degree 26 in $\mathbb{P}(1,1,5,7,13)$ with $-K_X^3 = \frac{2}{35}$. It has two singular points. One is a quotient singularity P of type $\frac{1}{5}(1,2,3)$ and the other is a quotient singularity Q of type $\frac{1}{7}(1,1,6)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,1,5), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,6)$,
- β is the Kawamata blow up with weights $(1,1,5)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{6}(1,1,5)$,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then $\mathcal{M} = |-K_X|$ by Corollary 0.3.8. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}$$

by Lemma 0.3.3.

Lemma 1.15.6. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point Q.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain the point Q. Then, it must consist of the point P. Let $\alpha_P : U_P \to X$ be the Kawamata blow up at the point P and let G_P be its exceptional divisor. Then, the proof is the same as that of Lemma 1.15.1. From the equivalences

$$\begin{cases} \alpha_P^*\big(-K_X\big) \sim_{\mathbb{Q}} S_{U_P} + \dfrac{1}{5}G_P, \\ \alpha_P^*\big(-K_X\big) \sim_{\mathbb{Q}} S_{U_P}^y + \dfrac{6}{5}G_P, \\ \alpha_P^*\big(-5K_X\big) \sim_{\mathbb{Q}} S_{U_P}^z, \\ \alpha_P^*\big(-7K_X\big) \sim_{\mathbb{Q}} S_{U_P}^t + \dfrac{2}{5}G_P, \\ \alpha_P^*\big(-13K_X\big) \sim_{\mathbb{Q}} S_{U_P}^w + \dfrac{3}{5}G_P,. \end{cases}$$

we see that the pull-backs of the rational functions 1, $\frac{y}{x}$, $\frac{zy}{x^6}$, $\frac{ty}{x^8}$ and $\frac{y^2w}{x^{15}}$ induce a birational map of U_P onto a hypersurface of degree 30 in $\mathbb{P}(1,1,6,8,15)$. □

The exceptional divisor E of the birational morphism α contains the singular point O of U that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

Lemma 1.15.7. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain the point O. Then, it must contain the point $\bar{P}$ whose image to X is the point P. Let $\beta_P : W_P \to U$ be the Kawamata blow up at the point $\bar{P}$ with weights $(1,2,3)$ and $F_P \cong \mathbb{P}(1,2,3)$ be its exceptional divisor. The proof of Lemma 1.15.6 implies that the log pair $(W_P, \frac{1}{n}\mathcal{M}_{W_P})$ is terminal. We have

$$\begin{cases} (\alpha\circ\beta_P)^*\big(-K_X\big) \sim_{\mathbb{Q}} S_{W_P} + \dfrac{1}{7}E_{W_P} + \dfrac{1}{5}F_P, \\ (\alpha\circ\beta_P)^*\big(-K_X\big) \sim_{\mathbb{Q}} S_{W_P}^y + \dfrac{1}{7}E_{W_P} + \dfrac{6}{5}F_P, \\ (\alpha\circ\beta_P)^*\big(-5K_X\big) \sim_{\mathbb{Q}} S_{W_P}^z + \dfrac{12}{7}E_{W_P}, \\ (\alpha\circ\beta_P)^*\big(-7K_X\big) \sim_{\mathbb{Q}} S_{W_P}^t + \dfrac{2}{5}F_P, \\ (\alpha\circ\beta_P)^*\big(-13K_X\big) \sim_{\mathbb{Q}} S_{W_P}^w + \dfrac{6}{7}E_{W_P} + \dfrac{3}{5}F_P,. \end{cases}$$

The equivalences imply

$$\begin{cases} (\alpha\circ\beta_P)^*\Big(\dfrac{y}{x}\Big) \in |S_{W_P}|, \\ (\alpha\circ\beta_P)^*\Big(\dfrac{yz}{x^6}\Big) \in |6S_{W_P}|, \\ (\alpha\circ\beta_P)^*\Big(\dfrac{yzt}{x^{14}}\Big) \in |14S_{W_P}|, \\ (\alpha\circ\beta_P)^*\Big(\dfrac{y^3zw}{x^{21}}\Big) \in |21S_{W_P}|, \end{cases}$$

and hence the pull-backs of rational functions 1, $\frac{y}{x}$, $\frac{yz}{x^6}$, $\frac{yzt}{x^{14}}$, and $\frac{y^3zw}{x^{21}}$ induce a birational map $\chi : W_P \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,6,14,21)$. Therefore, the divisor $-K_{W_P}$ is big.

The base locus of the pencil $|-K_{W_P}|$ consists of the irreducible curves C_{W_P} and $\bar{L}_{W_P}$, where $\bar{L}_{W_P}$ is the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on F_P. It follows from [21] that there is an antiflip $\zeta : W_P \dashrightarrow W_P'$ along the curve C_{W_P}. The divisor $-K_{W_P'}$ is nef and big. The singularities of the log pair $(W_P', \frac{1}{n}\mathcal{M}_{W_P'})$ are terminal because the rational map ζ is a log flop with respect to the log pair $(W_P, \frac{1}{n}\mathcal{M}_{W_P})$, which contradicts Theorem 0.2.4. □

Proposition 1.15.8. *The linear system $|-K_X|$ is the only Halphen pencil on X.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the singular point contained in the exceptional divisor F of the birational morphism β. Let $\gamma_1 : W \to Y$ be the Kawamata blow up at this point. Then, the pencil $|-K_W|$ is the proper transform of the pencil $|-K_X|$. Its base locus consists of the irreducible curve C_W. Because $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$, the inequality $-K_W \cdot C_W < 0$ implies that $\mathcal{M} = |-K_X|$ by Theorem 0.2.9.

Now, we suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ consists of the point $\hat{P}$ whose image to X is the point P. Let $\gamma : V \to Y$ be the Kawamata blow up at the point $\hat{P}$ with weights $(1,2,3)$. Then, the curve C_V is the only curve that intersects with $-K_V$ negatively. The proof of Lemma 1.15.6 implies that the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal. Therefore, there is an antiflip $\hat{\zeta} : V \dashrightarrow V'$ along the curve C_V. The log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ is also terminal. However, for some positive integer m, the linear system $|-mK_{V'}|$ induces an elliptic fibration onto $\mathbb{P}(1,1,6)$, which contradicts Theorem 0.2.4. $\square$

1.16. Case $\gimel = 51$, hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$.

The threefold X is a general hypersurface of degree 22 in $\mathbb{P}(1,1,4,6,11)$ with $-K_X^3 = \frac{1}{12}$. The singularities of X consist of one point that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, one point P that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and one point Q that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,4), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,5)$,
- β is the Kawamata blow up with weights $(1,1,4)$ at the singular point of the variety U contained in the exceptional divisor of the birational morphism α,
- η is an elliptic fibration.

By the generality of the hypersurface X, it can be given by an equation of the form

$$z^4t + z^3f_{10}(x,y,t) + z^2f_{14}(x,y,t,w) + zf_{18}(x,y,t,w) + f_{22}(x,y,t,w) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\xi : X \dashrightarrow \mathbb{P}^7$ be the rational map induced by the linear systems on the hypersurface X defined by the equations

$$\mu t + \sum_{i=0}^{6} \lambda_i x^i y^{6-i} = 0,$$

where $(\mu : \lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5 : \lambda_6) \in \mathbb{P}^7$. Then, the closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,6)$ and the normalization of a general fiber of ξ is an elliptic curve. We also have the following commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "\gamma"'] \arrow[dr, "\eta_0"] & \\ X \arrow[rr, dashed, "\xi"'] & & \mathbb{P}(1,1,6), \end{tikzcd}$$

where

- γ is the Kawamata blow up at the point P with weights $(1,1,3)$,

- η_0 is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains either the singular point of type $\frac{1}{2}(1,1,1)$ or a curve, then $\mathcal{M} = |-K_X|$ by Lemma 0.3.11 and Corollary 0.3.8. Therefore, Lemma 0.3.3 enables us to assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subseteq \{P, Q\}$

Lemma 1.16.1. *If $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P, Q\}$, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\pi_1 : V_1 \to W$ be the Kawamata blow up with weights $(1,1,5)$ at the point whose image to X is the point Q. Then, $\mathcal{M}_{V_1} \sim_{\mathbb{Q}} -nK_{V_1}$ by Lemma 0.2.6, the pencil $|-K_{V_1}|$ is the proper transform of the pencil $|-K_X|$, and the base locus of the pencil $|-K_{V_1}|$ consists of the irreducible curve C_{V_1}. Because $-K_{V_1} \cdot C_{V_1} < 0$, we obtain $\mathcal{M} = |-K_X|$ from Theorem 0.2.9. □

Lemma 1.16.2. *If $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$, then $\mathcal{M} = |-K_X|$.*

Proof. Let P_1 be the singular point of W contained in the exceptional divisor of γ. Then, $P_1 \in \mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ by Theorem 0.2.4, Lemmas 0.2.6, and 0.2.7.

Let $\pi_2 : V_2 \to W$ be the Kawamata blow up at the point P_1. Then, $M_{V_2} \sim_{\mathbb{Q}} -nK_{V_2}$ by Lemma 0.2.6, the pencil $|-K_{V_2}|$ is the proper transform of $|-K_X|$, and the base locus of $|-K_{V_2}|$ consists of the irreducible curve C_{V_2}. The inequality $-K_{V_2} \cdot C_{V_2} < 0$ implies $\mathcal{M} = |-K_X|$ by Theorem 0.2.9. □

Proposition 1.16.3. *The linear system $|-K_X|$ is the only pencil on X.*

Proof. Due to the previous arguments, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{Q\}$, which implies that $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ by Lemma 0.2.6. Hence, it follows from Theorem 0.2.4, Lemmas 0.2.7, and 0.2.3 that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the singular point of the threefold U that is contained in the exceptional divisor of α because the divisor $-K_U$ is nef and big.

Let P_2 be the singular point of Y contained in the exceptional divisor of the birational morphism β and let $\pi : V \to Y$ be the Kawamata blow up at the point P_2. Then, $M_V \sim_{\mathbb{Q}} -nK_V$ by Theorem 0.2.4, Lemmas 0.2.6, 0.2.7, and 0.2.3. On the other hand, the pencil $|-K_V|$ is the proper transform of the pencil $|-K_X|$ and the base locus of the pencil $|-K_V|$ consists of the irreducible curve C_V. Due to Theorem 0.2.9, the inequality $-K_V \cdot C_V < 0$ completes our proof. □

1.17. Case $\beth = 82$, hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$.

The threefold X is a general hypersurface of degree 36 in $\mathbb{P}(1,1,5,12,18)$ with $-K_X^3 = \frac{1}{30}$. Its singularities consist of two quotient singular points P and Q of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{6}(1,1,5)$, respectively. The hypersurface X can be given by the equation

$$z^7 y + \sum_{i=0}^{6} z^i f_{36-5i}\big(x, y, z, t\big) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i.

There is a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,5), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up of at the point Q with weights $(1,1,5)$,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve, then we obtain $\mathcal{M} = |-K_X|$ from Corollary 0.3.8. Therefore, we may assume that

$$\varnothing \neq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P, Q\Big\}$$

due to Lemma 0.3.3.

The exceptional divisor of the birational morphism α contains a singular point O of the threefold U that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

Lemma 1.17.1. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\alpha_O : U_O \to U$ be the Kawamata blow up at the point O. Then, the pencil $|-K_{U_O}|$ is the proper transform of the pencil $|-K_X|$. Its base locus consists of C_{U_O}. Furthermore, $-K_{U_O} \cdot C_{U_O} < 0$ and $\mathcal{M}_{U_O} \sim_{\mathbb{Q}} -nK_{U_O}$. Therefore, Theorem 0.2.9 implies the statement. $\square$

Therefore, if the set the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point Q, then $\mathcal{M} = |-K_X|$, and hence we may assume that $P \in \mathbb{CS}(X, \frac{1}{n}\mathcal{M})$.

Let $\pi : W \to X$ be the Kawamata blow up at the point P with weights $(1,2,3)$ and $E \cong \mathbb{P}(1,2,3)$ be its exceptional divisor. Then, the exceptional divisor E contains two singular points P_1 and P_2 that are quotient singularities of the threefold W of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$, respectively. Let L be the unique curve of the linear system $|\mathcal{O}_{\mathbb{P}(1,\,2,\,3)}(1)|$ on the surface E.

The linear system $|-K_W|$ is the proper transform of $|-K_X|$ and its base locus consists of the irreducible curves L_W and C_W. The divisor $-K_W$ is not nef because $-K_W \cdot C_W < 0$. The curve C_W is the only curve that has negative intersection with the divisor $-K_W$ because $-K_W \cdot L_W > 0$.

Let $\mathcal{D}$ be the proper transform of the linear system on the threefold X that is cut by

$$h_{21}\big(x,y\big) + zyh_{15}\big(x,y\big) + zty^3h_1\big(x,y\big) + z^2y^2h_9\big(x,y\big) +$$
$$+ z^3y^3h_3\big(x,y\big) + ty^2h_7\big(x,y\big) + h_0wy^3 = 0$$

where h_i is a homogeneous polynomial of degree i. Then, $\mathcal{D} \sim_{\mathbb{Q}} -21K_W$ but $\mathcal{D}$ induces a birational map $\gamma : W \dashrightarrow X'$ such that X' is a hypersurface of degree 42 in $\mathbb{P}(1,1,6,14,21)$ with canonical singularities (see the proof of Theorem 5.5.1 in [7]).

There is an antiflip $\xi : W \dashrightarrow W'$ along the curve L_W. The divisor $-K_{W'}$ is nef and big and the linear system $|-mK_{W'}|$ induced a birational morphism $\beta : W' \to X'$ for some natural number $m \gg 0$. Therefore, we have a commutative diagram

$$\begin{tikzcd}
& W \arrow[dl, "\pi"'] \arrow[rr, dashed, "\xi"] \arrow[drrr, dashed, "\gamma"] & & W' \arrow[dr, "\beta"] & \\
X \arrow[rrrr, dashed, "\omega"] \arrow[dr, hook] & & & & X' \arrow[dl, hook'] \\
& \mathbb{P}\big(1,1,5,12,18\big) \arrow[rr, dashed, "{(x,\ y,\ zy,\ ty^2,\ wy^3)}"] & & \mathbb{P}\big(1,1,6,14,21\big), &
\end{tikzcd}$$

where ω is the rational map induced by the linear system $\pi(\mathcal{D})$.

Lemma 1.17.2. *The log pair $(W, \frac{1}{n}\mathcal{M}_W)$ is not terminal.*

Proof. Suppose that the log pair $(W, \frac{1}{n}\mathcal{M}_W)$ is terminal. Then, the log pair $(W, \lambda\mathcal{M}_W)$ is terminal for some rational number $\lambda > \frac{1}{n}$. We have $\mathcal{M}_{W'} \sim_{\mathbb{Q}} -nK_{W'}$, but ξ is a log flip for $(W, \lambda\mathcal{M}_W)$. Therefore, the singularities of the log pair $(W', \frac{1}{n}\mathcal{M}_{W'})$ are terminal, which contradicts Theorem 0.2.4. $\square$

Lemma 1.17.3. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_i, then $\mathcal{M} = |-K_X|$.*

Proof. Let $\alpha_i : V_i \to W$ be the Kawamata blow up at the point P_i with weights $(1, i, 1)$ and F be the exceptional divisor of the birational morphism α_i. The pencil $|-K_{V_i}|$ is the proper transform of the pencil $|-K_X|$. Its base locus of the pencil $|-K_{V_i}|$ consists of two irreducible curves C_{V_i} and L_{V_i}.

On a general surface D in $|-K_{V_i}|$, we have

$$C_{V_i}^2 = -\frac{7}{6},\ L_{V_1}^2 = -\frac{4}{3},\ L_{V_2}^2 = -1,\ C_{V_i} \cdot L_{V_i} = 1.$$

The surface D is normal and the intersection form of the curves C_{V_i} and L_{V_i} on the surface D is negative-definite. On the other hand, we have

$$\mathcal{M}_{V_i}\Big|_D \equiv -nK_{V_i}\Big|_D \equiv nC_{V_i} + nL_{V_i}.$$

Therefore, $\mathcal{M} = |-K_X|$ by Theorem 0.2.9. □

From now, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P, Q\}$.

Let $\bar{Q}$ be the point of the threefold W such that $\pi(\bar{Q}) = Q$. Then,

$$\mathbb{CS}\Big(W, \frac{1}{n}\mathcal{M}_W\Big) = \Big\{\bar{Q}\Big\}$$

by Lemmas 1.17.2, 1.17.3, and 0.2.7.

Let $\gamma : V \to W$ be the blow up of $\bar{Q}$ with weights $(1, 1, 5)$ and F be its exceptional divisor. Then, the base locus of the pencil $|-K_V|$ consists of two irreducible curves C_V and L_V.

The arguments used in the proof of Lemma 1.17.3 together with Lemma 0.2.7 imply that either the singularities of the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ are terminal or $\mathcal{M} = |-K_X|$. We may assume that the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal, which implies that the log pair $(V, \lambda\mathcal{M}_V)$ is still terminal for some rational number $\lambda > \frac{1}{n}$.

Let $\mathcal{L}$ be the proper transform on the threefold V of the linear system

$$\mu_0 x^6 + \mu_1 y^6 + \mu_2 yz = 0,$$

where $(\mu_0 : \mu_1 : \mu_2) \in \mathbb{P}^2$. Then, $\mathcal{L} \sim_{\mathbb{Q}} -6K_V$ and the base locus of the linear system $\mathcal{L}$ consists of the curve C_V. So, the curve C_V is the only curve having negative intersection with $-K_V$.

The linear system $\mathcal{L}$ induces a rational map $\upsilon : V \dashrightarrow \mathbb{P}(1, 1, 6)$ whose general fiber is an elliptic curve. Moreover, it follows from [21] that there is a log flip $\chi : V \dashrightarrow V'$ along the curve C_V with respect to the log pair $(V, \lambda\mathcal{M}_V)$. The log pair $(V', \frac{1}{n}\chi(\mathcal{M}_V))$ is terminal and

$$\chi\big(\mathcal{M}_V\big) \equiv -nK_{V'},$$

which implies that $-K_{V'}$ is nef. Therefore, it follows from Theorem 3.1.1 in [14] that the linear system $|-mK_{V'}|$ is base-point-free for some natural number $m \gg 0$. However, the equivalence $\chi(\mathcal{L}) \equiv -6K_{V'}$ implies that the linear system $|-mK_{V'}|$ induces an elliptic fibration, which contradicts Theorem 0.2.4.

Consequently, we have proved

Proposition 1.17.4. *The linear system $|-K_X|$ is a unique Halphen pencil on X.*

Part 2. Fano threefold hypersurfaces with a single Halphen pencil.

2.1. Case $\gimel = 19$, hypersurface of degree 12 in $\mathbb{P}(1,2,3,3,4)$.

The threefold X is a general hypersurface of degree 12 in $\mathbb{P}(1,2,3,3,4)$ with $-K_X^3 = \frac{1}{6}$. It has seven singular points. Four points P_1, P_2, P_3, P_4 of them are quotient singularities of type $\frac{1}{3}(1,2,1)$ and the others are quotient singularities of type $\frac{1}{2}(1,1,1)$.

For each point P_i, we have the following diagram:

$$\begin{tikzcd} & Y_i \arrow[dl, "\pi_i"'] \arrow[dr, "\eta_i"] & \\ X \arrow[rr, dashed, "\psi_i"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where

- ψ_i is a projection,
- π_i is the Kawamata blow up at the point P_i with weights $(1,2,1)$,
- η_i is an elliptic fibration.

It follows from Corollary 0.3.7 and Lemmas 0.3.3, 0.3.10 that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subseteq \Big\{P_1, P_2, P_3, P_4\Big\}.$$

Lemma 2.1.1. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ cannot consist of a single point.*

Proof. Suppose that it contains only the point P_i. Then, the set $\mathbb{CS}(Y_i, \frac{1}{n}\mathcal{M}_{Y_i})$ must contain the singular point O_i contained in the exceptional divisor E_i of the birational morphism π_i. Let $\alpha_i : V_i \to Y_i$ be the Kawamata blow up at the point O_i. We consider the linear system $\mathcal{D}$ on X defined by the equations

$$\lambda_0 x^4 + \lambda_1 x^2 y + \lambda_2 w = 0,$$

where $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$. The base locus of the linear system $\mathcal{D}_{V_i}$ consists of the irreducible curve $\hat{C}_{V_i}$ whose image to X is the base curve of $\mathcal{D}$. A general surface D_i in $\mathcal{D}_{V_i}$ is normal and $\hat{C}_{V_i}^2 = -\frac{1}{6}$ on the surface D_i, which implies $\mathcal{M} = \mathcal{D}$. It is a contradiction since $\mathcal{D}$ is not a pencil. $\square$

Proposition 2.1.2. *The linear system $|-2K_X|$ is the only Halphen pencil on X.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the points P_i and P_j, where $i \neq j$. Then, the set $\mathbb{CS}(Y_i, \frac{1}{n}\mathcal{M}_{Y_i})$ must contain the singular point Q_j whose image to X is the point P_j. Let $\beta_j : W_j \to Y_i$ be the Kawamata blow up at the point Q_j. Then, we see that the linear system $|-2K_{W_j}|$ is the proper transform of the pencil $|-2K_X|$. The base locus of the pencil $|-2K_{W_j}|$ consists of the irreducible curve C_{W_j} whose image to X is the base curve of $|-2K_X|$. A general surface D in $|-2K_{W_j}|$ is normal and $C_{W_j}^2 = -\frac{1}{3}$ on the surface D, which implies $\mathcal{M}_{W_j} = |-2K_{W_j}|$. Therefore, $\mathcal{M} = |-2K_X|$. $\square$

2.2. Cases $\gimel = 23$ and 44.

For the case $\gimel = 23$, let X be a general hypersurface of degree 14 in $\mathbb{P}(1,2,3,4,5)$ with $-K_X^3 = \frac{7}{60}$. It has three quotient singularities of type $\frac{1}{2}(1,1,1)$, one quotient singularity of type $\frac{1}{3}(1,2,1)$, one quotient singularity P of type $\frac{1}{5}(1,2,3)$, and one quotient singularity Q of type $\frac{1}{4}(1,3,1)$.

We have the following elliptic fibration:

$$\begin{tikzcd}
 & W \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] \arrow[rrrd, "\eta"] & & & \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & \mathbb{P}(1,2,3), \\
 & X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1, 2, 3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1, 3, 1)$,
- β_Q is the Kawamata blow up with weights $(1, 3, 1)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1, 2, 3)$ at the point whose image to X is the point P,
- η is an elliptic fibration.

Because we mainly consider the pencil $|-2K_X|$, we let D be a general surface in $|-2K_X|$.

It follows from Corollary 0.3.8 and Lemmas 0.3.3, 0.3.10, and 0.3.11 that we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subset \{P, Q\}$.

Lemma 2.2.1. *If the log pair $(X, \frac{1}{n}\mathcal{M})$ is terminal at the point P, then $\mathcal{M} = |-2K_X|$.*

Proof. Suppose that the log pair $(X, \frac{1}{n}\mathcal{M})$ is terminal at the point P. Then, the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains the quotient singular point Q_1 of type $\frac{1}{2}(1, 1, 1)$ on U_Q contained in the exceptional divisor E of α_Q. Let $\gamma_Q : W_Q \to U_Q$ be the Kawamata blow up at the singular point Q_1 with weights $(1, 1, 1)$.

Then, $|-2K_{W_Q}|$ is the proper transform of the pencil $|-2K_X|$ and the base locus of the pencil $|-2K_{W_Q}|$ consists of the irreducible curve C_{W_Q}. We can easily check $D_{W_Q} \cdot C_{W_Q} = -4K^3_{W_Q} < 0$. Hence, we have $\mathcal{M} = |-2K_X|$ by Theorem 0.2.9 since $\mathcal{M}_{W_Q} \sim_{\mathbb{Q}} -nK_{W_Q}$ by Lemma 0.2.6. □

Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P. Let F be the exceptional divisor of the birational morphism α_P. Then, it contains two singular points O_1 and P_1 of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 2, 1)$, respectively.

Lemma 2.2.2. *The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ cannot contain the point O_1.*

Proof. Suppose that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point O_1. Let $\gamma_P : W_P \to U_P$ be the Kawamata blow up at the point O_1 with weights $(1, 1, 1)$. Let G_1 be the exceptional divisor of the birational morphism γ_P. Then, we have

$$\mathcal{M}_{W_P} \sim_{\mathbb{Q}} -nK_{W_P} \sim_{\mathbb{Q}} \gamma_P^*(-nK_{U_P}) - \frac{n}{2}G_1.$$

Let $\mathcal{B}$ be the proper transform of the linear system $|-3K_X|$ by the birational morphism $\alpha_P \circ \gamma_P$. We then see that $\mathcal{B} \sim_{\mathbb{Q}} \gamma_P^*(-3K_{U_P}) - \frac{1}{2}G_1$. The base curve of $\mathcal{B}$ consists of the proper transform C_{W_P}. Furthermore, $B \cdot C_{W_P} = \frac{1}{12}$, where B is a general surface in $\mathcal{B}$. However, we obtain a contradictory inequalities

$$\begin{aligned} -\frac{n^2}{4} &= (\gamma_P^*(-3K_{U_P}) - \frac{1}{2}G_1) \cdot (\gamma_P^*(-nK_{U_P}) - \frac{n}{2}G_1)^2 \\ &= (\gamma_P^*(-3K_{U_P}) - \frac{1}{2}G_1) \cdot M_1 \cdot M_2 \geq 0, \end{aligned}$$

where M_1 and M_2 are general surfaces in $\mathcal{M}_{W_P}$. □

Lemma 2.2.3. *If the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1, then $\mathcal{M} = |-2K_X|$.*

Proof. The proof of Lemma 2.2.1 implies the statement. $\square$

Proposition 2.2.4. *If $\gimel = 23$, then the linear system $|-2K_X|$ is a unique Halphen pencil on X.*

Proof. By the previous arguments, we may assume that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ consists of the single point O such that $\alpha_P(O) = Q$. The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must contain the singular point in the exceptional divisor of the birational morphism β_Q. Then, the proof of Lemma 2.2.1 completes the proof. $\square$

From now on, we consider the case $\gimel = 44$. The threefold X is a general hypersurface of degree 20 in $\mathbb{P}(1,2,5,6,7)$ with $-K_X^3 = \frac{1}{21}$. The singularities of the hypersurface X consist of three quotient singular point of type $\frac{1}{2}(1,1,1)$, one quotient singular point P of type $\frac{1}{7}(1,2,5)$, and one quotient singular point Q of type $\frac{1}{6}(1,5,1)$.

We have the following elliptic fibration:

$$\begin{tikzcd}
 & W \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1,2,5), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,2,5)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,5,1)$,
- β_Q is the Kawamata blow up with weights $(1,5,1)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1,2,5)$ at the point whose image to X is the point P,
- η is an elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation of degree 20 as follows:

$$w^2t + wf_{13}(x,y,z,t) + f_{20}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Consider the linear subsystem of the linear system $|-6K_X|$ defined by equations

$$\lambda_0 t + \lambda_1 y^3 + \lambda_2 y^2x^2 + \lambda_3 yx^4 + \lambda_4 x^6 = 0,$$

where $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) \in \mathbb{P}^4$. It gives us a rational map $\xi : X \dashrightarrow \mathbb{P}^4$ that is defined in the outside of the point P. The closure of the image of the rational map ξ is the surface $\mathbb{P}(1,1,3)$, which can be identified with a cone over a smooth rational curve of degree 3 in $\mathbb{P}^3$. Moreover, the normalization of a general fiber of the rational map ξ is an elliptic curve. Therefore, we have another elliptic fibration as follows:

$$\begin{tikzcd}
U_P \arrow[d, "\alpha_P"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta_0"] \\
X \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,1,3),
\end{tikzcd}$$

where

- γ is the Kawamata blow up with weights $(1,2,3)$ at the singular point of U_P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_P,
- η_0 is an elliptic fibration.

It follows from Lemmas 0.3.3 and 0.3.10 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subseteq \{P, Q\}$.

The only different part from the proof for the case $\daleth = 23$ is Lemma 2.2.3. Therefore, it is enough to show that if the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the quotient singularity P_1 of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_P, then $\mathcal{M} = |-2K_X|$. Suppose that the set contains the point P_1. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains a singular point. The exceptional divisor F of the birational morphism γ contains two singular points P_1' and P_2' of Y that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$.

By Lemma 2.2.1, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) \subset \{P_1', P_2'\}$.

Lemma 2.2.5. *The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ cannot contain the point P_1'.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point P_1'. Then, we consider the Kawamata blow up $\sigma_1 : Y_1 \to Y$ at the point P_1' with weights $(1,1,1)$. Let $\mathcal{T}$ be the linear subsystem of the linear system $|-6K_X|$ defined by the equations

$$\lambda_0 t + \lambda_1 x^6 + \lambda_2 x^4 y + \lambda_3 x^2 y^2 = 0,$$

where $(\lambda_0 : \lambda_1 : \lambda_2 : \lambda_3) \in \mathbb{P}^3$. Then, for a general surface T in the linear system $\mathcal{T}$, we have

$$T_{Y_1} \sim_{\mathbb{Q}} (\alpha_P \circ \gamma \circ \sigma_1)^*(-6K_X) - \frac{6}{7}(\gamma \circ \sigma_1)^*(E) - \frac{6}{5}\sigma_1^*(F) - G,$$

where E and G are the exceptional divisors of the birational morphisms α_P and σ_1, respectively. In addition, we see

$$F_{Y_1} \sim_{\mathbb{Q}} (\gamma \circ \sigma_1)^*(E) - \frac{3}{5}\sigma_1^*(F) - \frac{1}{2}G.$$

Let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,5)}(1)|$ on the surface E. The base locus of the proper transform $\mathcal{T}_{Y_1}$ contains the irreducible curve $\tilde{C}_{Y_1}$ and the irreducible curve L_{Y_1} such that

$$T_{Y_1} \cdot S_{Y_1} = \tilde{C}_{Y_1} + L_{Y_1}, \quad T_{Y_1} \cdot E_{Y_1} = 2L_{Y_1}.$$

The surface T_{Y_1} is normal and

$$\tilde{C}_{Y_1}^2 = L_{Y_1}^2 = -\frac{7}{4}, \quad \tilde{C}_{Y_1} \cdot L_{Y_1} = \frac{5}{4}$$

on the surface T_{Y_1}. The intersection form of the curves $\tilde{C}_{Y_1}$ and L_{Y_1} is negative-definite on the surface T_{Y_1}. Because $\mathcal{M}_{Y_1}|_{T_{Y_1}} \equiv nS_{Y_1}|_{T_{Y_1}} \equiv n\tilde{C}_{Y_1} + nL_{Y_1}$, we obtain an contradictory identity $\mathcal{M} = \mathcal{T}$ from Theorem 0.2.9. □

Proposition 2.2.6. *If $\daleth = 44$, then the linear system $|-2K_X|$ is a unique Halphen pencil on X.*

Proof. By Lemma 2.2.5, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point P_2'. Let $\sigma_2 : Y_2 \to Y$ be the Kawamata blow up at the point P_2'. Then, the pencil $|-2K_{Y_2}|$ is the proper transform of the pencil $|-2K_X|$ and its base locus consists of the irreducible curve C_{Y_2}. We can easily check $-K_{Y_2} \cdot C_{Y_2} = -2K_{Y_2}^3 < 0$. Hence, we obtain the identity $\mathcal{M} = |-2K_X|$ from Theorem 0.2.9 because $\mathcal{M}_{Y_2} \sim_{\mathbb{Q}} -nK_{Y_2}$ by Lemma 0.2.6. □

2.3. Cases $\daleth = 27$, 42, and 68.

The aim of this section is to prove the following:

Proposition 2.3.1. *If $\daleth = 27$, 42, or 68, then the linear system $|-a_1K_X|$ is a unique Halphen pencil on X.*

We first consider the case $\daleth = 68$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 28 in $\mathbb{P}(1,3,4,7,14)$ with $-K_X^3 = \frac{1}{42}$. The singularities consist of two quotient singular points P and Q of type $\frac{1}{7}(1,3,4)$, one point of type $\frac{1}{3}(1,1,2)$, and one point of type $\frac{1}{2}(1,1,1)$.

We have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\sigma_Q"'] \arrow[dr, "\sigma_P"] \arrow[rrrd, "\eta"] & & & \\
Y_P \arrow[dr, "\pi_P"'] & & Y_Q \arrow[dl, "\pi_Q"] & & \mathbb{P}(1,3,4), \\
 & X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,3,4)$,
- π_Q is the Kawamata blow up at the point Q with weights $(1,3,4)$,
- σ_Q is the Kawamata blow up with weights $(1,3,4)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,3,4)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ is nonempty by Theorem 0.2.4. We may also assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}$$

due to Lemma 0.3.3 and Corollary 0.3.7. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P. The exceptional divisor $E \cong \mathbb{P}(1,3,4)$ of the birational morphism π_P contains two quotient singular points O_1 and O_2 of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively.

Lemma 2.3.2. *The set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ cannot contain the point O_2.*

Proof. Suppose so. We then consider the Kawamata blow up $\alpha : U \to Y_P$ at the point O_2 with weights $(1,2,1)$. Let $\mathcal{D}$ be the proper transform of the linear system $|-4K_X|$ on X by the birational morphism $\pi_P \circ \alpha$. Its base locus consists of the irreducible curve $\bar{C}_U$. For a general surface D_U in $\mathcal{D}$, we have

$$D_U \sim_{\mathbb{Q}} (\pi_P \circ \alpha)^*(-4K_X) - \frac{4}{7}\alpha^*(E) - \frac{1}{3}F,$$

where F is the exceptional divisor of α. The surface D_U is normal. We also have

$$S_U \sim_{\mathbb{Q}} (\pi \circ \sigma)^*(-K_X) - \frac{1}{7}\alpha^*(E) - \frac{1}{3}F$$

and $S_U \cdot D_U = \bar{C}_U$. On the normal surface D_U, the curve $\bar{C}_U$ has negative self-intersection number $\bar{C}_U^2 = -\frac{5}{42}$. However, we have

$$\mathcal{M}_U\Big|_{D_U} \equiv -nK_U\Big|_{D_U} \equiv n\bar{C}_U.$$

Therefore, Theorem 0.2.9 implies $\mathcal{M} = |-4K_X|$. It is a contradiction because the linear system $|-4K_X|$ is not a pencil. □

Lemma 2.3.3. *If the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the point O_1, then $\mathcal{M} = |-3K_X|$.*

Proof. Let $\beta : W \to Y_P$ be the Kawamata blow up at the point O_1 with weights $(1,3,1)$. We consider the proper transform $\mathcal{P}$ of the pencil $|-3K_X|$ by the birational morphism $\pi_P \circ \beta$. Its base locus consists of the irreducible curve C_W. Because the curve C_W has negative self-intersection on a general surface in $\mathcal{P}$ and a general surface in $\mathcal{P}$ is normal, we can obtain $\mathcal{M} = |-3K_X|$ from Theorem 0.2.9. □

Therefore, we may assume that the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ consists of the single point Q_1. The exceptional divisor of σ_Q contains two singular points Q_2 and Q_3 of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively. Then, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ is nonempty. Furthermore, it must contain either the point Q_2 or the point Q_3. However, the proof of Lemma 2.3.2 shows it cannot contain the point Q_3. Also, Lemma 2.3.3 shows that $\mathcal{M} = |-3K_X|$ if the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point Q_2.

Following the same way, we can also conclude that $\mathcal{M} = |-3K_X|$ if the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point Q.

In the case $\gimel = 42$, the hypersurface X is given by a general quasihomogeneous equation of degree 20 in $\mathbb{P}(1,2,3,5,10)$ with $-K_X^3 = \frac{1}{15}$. The singularities consist of two quotient singular points P and Q of type $\frac{1}{5}(1,2,3)$, one point of type $\frac{1}{3}(1,2,1)$, and two points of type $\frac{1}{2}(1,1,1)$.

We have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[ld, "\sigma_Q"'] \arrow[rd, "\sigma_P"] \arrow[rrrd, "\eta"] & & & \\
Y_P \arrow[rd, "\pi_P"'] & & Y_Q \arrow[ld, "\pi_Q"] & & \mathbb{P}(1,2,3), \\
 & X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,2,3)$,
- π_Q is the Kawamata blow up at Q with weights $(1,2,3)$,
- σ_Q is the Kawamata blow up with weights $(1,2,3)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,2,3)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

Using the exactly same method as in the case $\gimel = 68$, one can show that $\mathcal{M} = |-2K_X|$.

From now, we consider the case $\gimel = 27$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 15 in $\mathbb{P}(1,2,3,5,5)$ with $-K_X^3 = \frac{1}{10}$. The singularities consist of three quotient singular points P_1, P_2, and P_3 of type $\frac{1}{5}(1,2,3)$ and one point of type $\frac{1}{2}(1,1,1)$.

And we have a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[ld, "\pi"'] \arrow[rd, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3),
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow ups at the points P_1, P_2 and P_3 with weights $(1,2,3)$,
- η is an elliptic fibration.

Even though three singular points are involved in this case, the same method as in the previous cases can be applied to obtain $\mathcal{M} = |-2K_X|$.

2.4. Case $\gimel = 32$, hypersurface of degree 16 in $\mathbb{P}(1,2,3,4,7)$.

The hypersurface X is given by a general quasihomogeneous polynomial of degree 16 in $\mathbb{P}(1,2,3,4,7)$ with $-K_X^3 = \frac{2}{21}$. The singularities of the threefold X consist of four quotient

singular points of type $\frac{1}{2}(1,1,1)$, one quotient singular point of type $\frac{1}{3}(1,2,1)$, and one quotient singular point P of type $\frac{1}{7}(1,3,4)$. There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & Y \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,3,4)$,
- β is the Kawamata blow up with weights $(1,1,3)$ at the singular point Q of the variety U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α,
- η is an elliptic fibration.

The hypersurface X can be given by the quasihomogeneous equation

$$w^2y+wf_9\big(x,y,z,t\big)+f_{16}\big(x,y,z,t\big)=0$$

where f_9 and f_{16} are quasihomogeneous polynomials of degrees 9 and 16, respectively. Let D be a general surface in $|-2K_X|$. It is cut out on the threefold X by the equation

$$\lambda x^2+\mu y=0,$$

where $(\lambda:\mu)\in\mathbb{P}^1$. The surface D is irreducible and normal. The base locus of the pencil $|-2K_X|$ consists of the curve C, which implies that $C=D\cdot S$.

If the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{3}(1,2,1)$, we obtain $\mathcal{M}=|-2K_X|$ from Lemma 0.3.11. It then follows from Corollary 0.3.7 and Lemma 0.3.3 that

$$\mathbb{CS}\Big(X,\frac{1}{n}\mathcal{M}\Big)=\Big\{P\Big\}.$$

Furthermore, the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 0.2.4 because $-K_U$ is nef and big.

The exceptional divisor $E\cong\mathbb{P}(1,3,4)$ of the birational morphism α contains two singular points O and Q that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively. Let L be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$ on the surface E. Let F be the exceptional divisor of β. It contains a singular point Q_1 that is quotient singularity of type $\frac{1}{3}(1,1,2)$.

Then, it follows from Lemma 0.2.7 that either $Q\in\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ or $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)=\{O\}$.

Lemma 2.4.1. *If the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ consists of the point Q, then $\mathcal{M}=|-2K_X|$.*

Proof. It follows from Lemma 0.2.6 that $\mathcal{M}_Y\sim_{\mathbb{Q}}-nK_Y$, which implies that every surface in the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η and the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1.

Let $\pi:V\to Y$ be the Kawamata blow up at the point Q_1 with weights $(1,1,2)$. Then, the transform D_V is normal but the base locus of the pencil $|-2K_V|$ consists of the irreducible curves C_V and L_V.

The intersection form of the curves C_V and L_V on the surface D_V is negative-definite because the curves C_V and L_V are components of a fiber of the elliptic fibration $\eta\circ\pi|_{D_V}$ that contains three irreducible components. On the other hand, we have

$$\mathcal{M}_V\Big|_{D_V}\equiv-nK_V\Big|_{D_V}\equiv nC_V+nL_V.$$

Therefore, it follows from Theorem 0.2.9 that $\mathcal{M}=|-2K_X|$. $\square$

From now on, we may assume that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O due to Lemma 0.2.7. Let $\gamma : W \to U$ be the Kawamata blow up at the point O with weights $(1,1,2)$ and G be the exceptional divisor of the birational morphism γ. Then, the surface $G \cong \mathbb{P}(1,1,2)$ and

$$\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W \sim_{\mathbb{Q}} \gamma^*(-nK_U) - \frac{n}{3}G \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-nK_X) - \frac{n}{7}\gamma^*(E) - \frac{n}{3}G.$$

In a neighborhood of the point P, the monomials x, z, and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 3$, and $\mathrm{wt}(z) = 4$. Then, in a neighborhood of the singular point P, the surface D can be given by equation

$$\lambda x^2 + \mu\Big(\epsilon_1 x^9 + \epsilon_2 z x^6 + \epsilon_3 z^2 x^3 + \epsilon_4 z^3 + \epsilon_5 t^2 x + \epsilon_6 t x^5 + \epsilon_7 t z x^2 + h_{16}(x,z,t) + \text{higher terms}\Big) = 0,$$

where $\epsilon_i \in \mathbb{C}$ and h_{16} is a quasihomogeneous polynomial of degree 16. In a neighborhood of the singular point O, the birational morphism α can be given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{7}},\ z = \tilde{z}^{\frac{3}{7}},\ t = \tilde{t}\tilde{z}^{\frac{4}{7}},$$

where $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are weighted local coordinates on the variety U in a neighborhood of the singular point O such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 2$, and $\mathrm{wt}(\tilde{t}) = 1$.

In a neighborhood of the point O, the surface E is given by $\tilde{z} = 0$, the surface D_U is given by

$$\lambda \tilde{x}^2 + \mu\Big(\epsilon_1 \tilde{x}^9 \tilde{z} + \epsilon_2 \tilde{z}\tilde{x}^6 + \epsilon_3 \tilde{z}\tilde{x}^3 + \epsilon_4 \tilde{z} + \epsilon_5 \tilde{t}^2 \tilde{x}\tilde{z} + \epsilon_6 \tilde{t}\tilde{x}^5 \tilde{z} + \epsilon_7 \tilde{t}\tilde{z}\tilde{x}^2 + \text{higher terms}\Big) = 0,$$

and the surface S_U is given by the equation $\tilde{x} = 0$.

In a neighborhood of the singular point of G, the birational morphism γ can be given by

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{3}},\ \tilde{z} = \bar{z}^{\frac{2}{3}},\ \tilde{t} = \bar{t}\bar{z}^{\frac{1}{3}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on the variety W in a neighborhood of the singular point of G such that $\mathrm{wt}(\bar{x}) = \mathrm{wt}(\bar{z}) = \mathrm{wt}(\bar{t}) = 1$. The surface G is given by the equation $\bar{z} = 0$, the proper transform D_W is given by

$$\lambda \bar{x}^2 + \mu\Big(\epsilon_1 \bar{x}^9 \bar{z}^3 + \epsilon_2 \bar{z}^2 \bar{x}^6 + \epsilon_3 \bar{z}\bar{x}^3 + \epsilon_4 + \epsilon_5 \bar{t}^2 \bar{x}\bar{z} + \epsilon_6 \bar{t}\bar{x}^5 \bar{z}^2 + \epsilon_7 \bar{t}\bar{z}\bar{x}^2 + \text{higher terms}\Big) = 0,$$

the proper transform S_W is given by the equation $\bar{x} = 0$, and the proper transform E_W is given by the equation $\bar{z} = 0$.

Let $\mathcal{P}$ be the proper transforms on the variety W of the pencil $|-2K_X|$. The curves C_W and L_W are contained in the base locus of the pencil $\mathcal{P}$. Moreover, easy calculations show that the base locus of the pencil $\mathcal{P}$ does not contain any other curve than C_W and L_W. We also have

$$\begin{cases} E_W \sim_{\mathbb{Q}} \gamma^*(E) - \dfrac{2}{3}F, \\ D_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\big(-2K_X\big) - \dfrac{2}{7}\gamma^*(E) - \dfrac{2}{3}G, \\ S_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\big(-K_X\big) - \dfrac{1}{7}\gamma^*(E) - \dfrac{1}{3}G, \end{cases}$$

Also, we have $C_W + L_W = S_W \cdot D_W$ and $2L_W = D_W \cdot E_W$.

The curves C_W and L_W can be considered as irreducible effective divisors on the normal surface D_W. Then, it follows from the equivalences above that

$$L_W^2 = -\frac{5}{8},\ C_W^2 = -\frac{7}{24},\ C_W \cdot L_W = \frac{3}{8},$$

which implies that the intersection form of C_W and L_W on D_W is negative-definite. Let M be a general surface of the linear system $\mathcal{M}_W$. Then,

$$M\Big|_{D_W} \equiv -nK_W\Big|_{D_W} \equiv nS_W\Big|_{D_W} \equiv nC_W + nL_W,$$

which implies that $\mathcal{M} = |-2K_X|$ by Theorem 0.2.9.

Consequently, we have proved

Proposition 2.4.2. *The linear system $|-2K_X|$ is the only Halphen pencil on X.*

2.5. Cases $\gimel = 33$ and 38

For the case $\gimel = 38$, let X be the hypersurface given by a general quasihomogeneous equation of degree 18 in $\mathbb{P}(1,2,3,5,8)$ with $-K_X^3 = \frac{1}{35}$. Then, the singularities of X consist of two singular points P and Q that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{8}(1,3,5)$, respectively, and two points of type $\frac{1}{2}(1,1,1)$.

We have the following commutative diagram:

$$\begin{tikzcd}
& & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[rrrd, "\eta"] & & & \\
& U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & \mathbb{P}(1,2,3), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \\
& X \arrow[rrrruu, dashed, "\psi"] & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,2,3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,3,5)$,
- β_Q is the Kawamata blow up with weights $(1,3,5)$ at the point whose image by the birational morphism α_P is the point Q,
- β_P is the Kawamata blow up with weights $(1,2,3)$ at the point whose image by the birational morphism α_Q is the point P,
- β_O is the Kawamata blow up with weights $(1,3,2)$ at the singular point O of the variety U_Q that is a quotient singularity of type $\frac{1}{5}(1,3,2)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,2,3)$ at the point whose image by the birational morphism $\alpha_Q \circ \beta_O$ is the point P,
- γ_O is the Kawamata blow up with weights $(1,3,2)$ at the singular point of the variety U_{PQ} that is a quotient singularity of type $\frac{1}{5}(1,3,2)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

By Lemma 0.3.3 and Corollary 0.3.7, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}.$$

The exceptional divisor E_P of the birational morphism α_P contains two quotient singular points P_1 and P_2 of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{2}(1,1,1)$, respectively.

Lemma 2.5.1. *If the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1, then $\mathcal{M} = |-2K_X|$.*

Proof. Suppose it contains the point P_1. Let $\beta_1 : W_1 \to U_P$ be the Kawamata blow up at the point P_1 with weights $(1,2,1)$. The pencil $|-K_{W_1}|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the irreducible curve C_{W_1}.

For a general surface D_{W_1} in $|-2K_{W_1}|$, we can easily check $D_{W_1} \cdot C_{W_1} = -4K_{W_1}^3 < 0$. Hence, we obtain $\mathcal{M} = |-2K_X|$ from Theorem 0.2.9 because $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} nD_{W_1}$ by Lemma 0.2.6. $\square$

Lemma 2.5.2. *The set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ cannot contain the point P_2.*

Proof. Suppose it contains the point P_2. Let $\beta_2 : W_2 \to U_P$ be the Kawamata blow up at the point P_2 with weights $(1,1,1)$. Also, let $\mathcal{D}_2$ be the proper transform of the linear system $|-3K_X|$ by the birational morphism $\alpha_P \circ \beta_2$. Its base locus consists of the irreducible curve $\bar{C}_{W_2}$. A general surface D_{W_2} in $\mathcal{D}_2$ is normal and the self-intersection $\bar{C}_{W_2}^2$ is negative on the surface D_{W_2}. Because $\mathcal{M}_{W_2}|_{D_{W_2}} \equiv -n\bar{C}_{W_2}$, we obtain an absurd equality $\mathcal{M} = |-3K_X|$ from Theorem 0.2.9. □

Meanwhile, the exceptional divisor $E \cong \mathbb{P}(1,3,5)$ of the birational morphism α_Q contains two singular points O and Q_1 of types $\frac{1}{5}(1,3,2)$ and $\frac{1}{3}(1,1,2)$, respectively. For the convenience, let L be the unique curve on the surface E contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,5)}(1)|$

Lemma 2.5.3. *If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains the point Q_1, then $\mathcal{M} = |-2K_X|$.*

Proof. Let $\pi_1 : V_1 \to U_Q$ be the Kawamata blow up at the point Q_1 with weights $(1,1,2)$. The pencil $|-2K_{V_1}|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the irreducible curves C_{V_1}, L_{V_1}, and a curve $\bar{L}$ on the exceptional divisor $F_Q \cong \mathbb{P}(1,1,2)$ of the birational morphism π_1 contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$.

Let D_{V_1} be a general surface in $|-2K_{V_1}|$. We see then

$$S_{V_1} \cdot D_{V_1} = C_{V_1} + L_{V_1} + \bar{L},\ E_{V_1} \cdot D_{V_1} = 2L_{V_1},\ F_Q \cdot D_{V_1} = 2\bar{L}.$$

Using the following equivalences

$$\begin{cases} E_{V_1} \sim_{\mathbb{Q}} \pi_1^*(E) - \dfrac{1}{3}F_Q, \\ S_{V_1} \sim_{\mathbb{Q}} \big(\alpha_Q \circ \pi_1\big)^*\Big(-K_X\Big) - \dfrac{1}{8}\pi_1^*(E) - \dfrac{1}{3}F_Q, \\ D_{V_1} \sim_{\mathbb{Q}} \big(\alpha_Q \circ \pi_1\big)^*\Big(-2K_X\Big) - \dfrac{2}{8}\pi_1^*(E) - \dfrac{2}{3}F_Q, \end{cases}$$

we obtain

$$C_{V_1}^2 = -\frac{37}{20},\ L_{V_1}^2 = -\frac{7}{20},\ \bar{L}^2 = -\frac{3}{4},\ C_{V_1} \cdot L_{V_1} = 0,\ C_{V_1} \cdot \bar{L} = 1,\ L_{V_1} \cdot \bar{L} = \frac{1}{4}$$

on the normal surface D_{V_1}. One can see that the intersection form of these three curves on D_{V_1} is negative-definite. Let M be a general surface in the linear system $\mathcal{M}_{V_1}$. Then,

$$M\Big|_{D_{V_1}} \equiv -nK_{V_1}\Big|_{D_{V_1}} \equiv nS_{V_1}\Big|_{D_{V_1}} \equiv nC_{V_1} + nL_{V_1} + n\bar{L},$$

which implies that $\mathcal{M} = |-2K_X|$ by Theorem 0.2.9. □

The exceptional divisor F_O of the birational morphism β_O contains two quotient singular points O_1 and O_2 of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$, respectively.

Lemma 2.5.4. *If the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the point O_1, then $\mathcal{M} = |-2K_X|$.*

Proof. Let $\sigma_1 : U_1 \to U_{QO}$ be the Kawamata blow up at the point O_1 with weights $(1,1,2)$. The pencil $|-2K_{U_1}|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the irreducible curves C_{U_1} and L_{U_1}.

Let D_{U_1} be a general surface in $|-2K_{U_1}|$. We see then

$$S_{U_1} \cdot D_{U_1} = C_{U_1} + L_{U_1},\ E_{U_1} \cdot D_{U_1} = 2L_{U_1}.$$

Using the same argument as in Lemma 2.5.3, one can see that the intersection form of these two curves on D_{U_1} is negative-definite, and hence $\mathcal{M} = |-2K_X|$. □

Lemma 2.5.5. *The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ cannot contain the point O_2.*

Proof. Let $\sigma_2 : U_2 \to U_{QO}$ be the Kawamata blow up at the point O_2 with weights $(1,1,1)$ and let $\mathcal{D}$ be the proper transform of the linear system $|-3K_X|$ by the birational morphism $\alpha_Q \circ \beta_O \circ \sigma_2$.

The base locus of the linear system $\mathcal{D}$ consists of the irreducible curve C_{U_2}. The same method as in Lemma 2.5.1 shows that $\mathcal{M} = |-3K_X|$, which is a contradiction. □

Proposition 2.5.6. *If $\gimel = 38$, then the linear system $|-2K_X|$ is the only Halphen pencil on X.*

Proof. Due to the previous lemmas, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{P, Q\Big\}.$$

Following the Kawamata blow ups $Y \to U_{QO} \to U_Q \to X$ and using Lemmas 2.5.3, 2.5.4, and 2.5.5, we can furthermore assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains one of singular points contained in the exceptional divisor of the birational morphism γ_P. In this case, Lemmas 2.5.1 and 2.5.2 imply the statement. □

From now, we consider the case $\gimel = 33$. The variety X is a general hypersurface of degree 17 in $\mathbb{P}(1,2,3,5,7)$ with $-K_X^3 = \frac{17}{210}$. The singularities of X consist of one quotient singularity of type $\frac{1}{2}(1,1,1)$, one point that is a quotient singularity of type $\frac{1}{3}(1,2,1)$, one point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$, and one point Q that is a quotient singularity of type $\frac{1}{7}(1,2,5)$.

We have a commutative diagram as follows:

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drrr, "\eta"] & & & \\
 & U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & & \mathbb{P}(1,2,3), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & & \\
 & X \arrow[uurrrr, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,2,3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,2,5)$,
- β_Q is the Kawamata blow up with weights $(1,2,5)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1,2,3)$ at the point whose image to X is the point P,
- β_O is the Kawamata blow up with weights $(1,2,3)$ at the quotient singular point O of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,2,3)$ at the point whose image to X is the point P,
- γ_O is the Kawamata blow up with weights $(1,2,3)$ at the quotient singular point of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

It follows from Lemma 0.3.10 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain the singular point of type $\frac{1}{2}(1,1,1)$. Moreover, Lemma 0.3.11 implies that if the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{3}(1,2,1)$, then $\mathcal{M} = |-2K_X|$. Therefore, due to Corollary 0.3.7 and Lemma 0.3.3,

we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}.$$

Proposition 2.5.7. *If $\daleth = 33$, then the linear system $|-2K_X|$ is the only Halphen pencil on X.*

The proof is almost same as the proof of the case $\daleth = 38$. Lemmas 2.5.1 and 2.5.2 work for the case $\daleth = 33$.

The exceptional divisor $E \cong \mathbb{P}(1,2,5)$ of the birational morphism α_Q contains two singular points O and Q_1 of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$, respectively. For the case $\daleth = 33$, Lemma 2.5.3 should be replaced by the following:

Lemma 2.5.8. *The set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ cannot contain the point Q_1.*

Proof. Suppose that it contains the point Q_1. Let $\pi_1 : V_1 \to U_Q$ be the Kawamata blow up at the point Q_1. Consider the linear system $\mathcal{T}$ on X cut out by the equations

$$\lambda_0 t + \lambda_1 x^5 + \lambda_2 x^3 y = 0,$$

where $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$. For a general surface T in $\mathcal{T}$, we have

$$T_{V_1} \sim_{\mathbb{Q}} (\alpha_Q \circ \pi_1)^*(-5K_X) - \frac{5}{7}(E) - \frac{1}{2}F_Q,$$

where F_Q is the exceptional divisor of the birational morphism π_1. The base locus of the proper transform $\mathcal{T}_{V_1}$ consists of the irreducible curve $\tilde{C}_{V_1}$. The surface T_{V_1} is normal and $\tilde{C}_{V_1}^2 < 0$ on T_{V_1}. Because $\mathcal{M}_{V_1}|_{T_{V_1}} \equiv nS_{V_1}|_{T_{V_1}} \equiv n\tilde{C}_{V_1}$, we obtain an contradictory identity $\mathcal{M} = \mathcal{D}$ from Theorem 0.2.9. ☐

The exceptional divisor F_O of the birational morphism β_O contains two quotient singular points O_1 and O_2 of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$, respectively. Lemma 2.5.4 should be also replaced by the following:

Lemma 2.5.9. *The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ cannot contain the point O_1.*

Proof. Suppose it contains the point O_1. Let $\sigma_1 : U_1 \to U_{QO}$ be the Kawamata blow up at the point O_1 with weights $(1,1,1)$. Let $\mathcal{D}$ be the proper transform of the linear system $|-3K_X|$ by the birational morphism $\alpha_Q \circ \beta_O \circ \sigma_1$. Its base locus consists of two irreducible curves. One is the curve $\bar{C}_{U_1}$ and the other is the proper transform L_{U_1} of the unique curve L in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,5)}(1)|$ on the surface E. For a general surface D_{U_1} in $\mathcal{D}$,

$$\begin{cases} E_{U_1} \sim_{\mathbb{Q}} (\beta_O \circ \sigma_1)^*(E) - \dfrac{3}{5}\sigma_1^*(F_O) - \dfrac{1}{2}G, \\ S_{U_1} \sim_{\mathbb{Q}} (\alpha_Q \circ \beta_O \circ \sigma_1)^*(-K_X) - \dfrac{1}{7}(\beta_O \circ \sigma_1)^*(E) - \dfrac{1}{5}\sigma_1^*(F_O) - \dfrac{1}{2}G, \\ D_{U_1} \sim_{\mathbb{Q}} (\alpha_Q \circ \beta_O \circ \sigma_1)^*(-3K_X) - \dfrac{3}{7}(\beta_O \circ \sigma_1)^*(E) - \dfrac{3}{5}\sigma_1^*(F_O) - \dfrac{1}{2}G, \end{cases}$$

where G is the exceptional divisor of σ_1. We see also

$$S_{U_1} \cdot D_{U_1} = \bar{C}_{U_1} + L_{U_1}, \; E_{U_1} \cdot D_{U_1} = 2L_{U_1}.$$

Using the same argument as in Lemma 2.5.3, one can see that the intersection form of these two curves on D_{U_1} is negative-definite, and hence $\mathcal{M} = |-3K_X|$. It is a contradiction. ☐

Finally, we complete the proof of Proposition 2.5.7 by replacing Lemma 2.5.5 by the following:

Lemma 2.5.10. *If the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the point O_2, then $\mathcal{M} = |-2K_X|$.*

Proof. Let $\sigma_2 : U_2 \to U_{QO}$ be the Kawamata blow up at the point O_2 with weights $(1,2,1)$. The pencil $|-2K_{U_2}|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the irreducible curve C_{U_2}. Because $\mathcal{M}_{U_2} \sim_{\mathbb{Q}} -nK_{U_2}$ and $-K_{U_2} \cdot C_{U_2} < 0$, we obtain $\mathcal{M} = |-2K_X|$ from Theorem 0.2.9. ☐

2.6. Cases $\gimel = 37,\ 39,\ 52,\ 59,\ 73$, and 78.

Suppose that $\gimel \in \{37, 39, 52, 59, 73\}$. Then, the threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ always contains the point $O = (0 : 0 : 0 : 1 : 0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, a_2, a_3 - a_2)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point O with weights $(1, a_2, a_3 - a_2)$,
- η is an elliptic fibration.

Proposition 2.6.1. *If* $\gimel \in \{37, 39, 52, 59, 73\}$, *then* $\mathcal{M} = | - a_1 K_X|$.

Proof. We may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point O by Lemmas 0.3.3, 0.3.10, 0.3.11 and Corollary 0.3.7.

Let P be the singular point contained in the exceptional divisor of the birational morphism π that is a quotient singular point of type $\frac{1}{a_2}(1, a_2 - 1, 1)$. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point P by Theorem 0.2.4 and Lemma 0.2.7.

Let $\alpha : W \to Y$ be the Kawamata blow up at the point P with weights $(1, a_2 - 1, 1)$, and $\mathcal{P}$ be the proper transforms of the pencil $| - a_1 K_X|$ by the birational morphism $\pi \circ \alpha$. Then,

$$\mathcal{P} \sim_{\mathbb{Q}} -a_1 K_W,\ \mathcal{M}_W \sim_{\mathbb{Q}} -n K_W.$$

One can easily check that the base locus of the pencil $\mathcal{P}$ consists of the irreducible curve C_W. A general surface D in the pencil $\mathcal{P}$ is normal and the inequality $C_W^2 < 0$ holds on the surface D, which implies that $\mathcal{M} = | - a_1 K_X|$ by Theorem 0.2.9. □

Proposition 2.6.2. *If* $\gimel = 78$, *then* $\mathcal{M} = | - a_1 K_X|$.

Proof. The only different thing from the proof of Proposition 2.6.1 is that the exceptional divisor E contains another singular point Q. It is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$. For the case $\gimel = 78$, we have to consider the case when $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) = \{Q\}$. In this case, applying the method in Proposition 2.6.1 to the Kawamata blow up at the point Q and the linear system $| - a_2 K_X|$, we can easily obtain $\mathcal{M} = | - a_2 K_X|$. However, it is a contradiction because the linear system $| - a_2 K_X|$ is not a pencil. Therefore, the case when $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) = \{Q\}$ never happens. □

2.7. Cases $\gimel = 40$ and 61.

For the case $\gimel = 40$, let X be the hypersurface given by a general quasihomogeneous equation of degree 19 in $\mathbb{P}(1, 3, 4, 5, 7)$ with $-K_X^3 = \frac{19}{420}$. Then, the singularities of X consist of one quotient singular point P of type $\frac{1}{7}(1, 3, 4)$, one quotient singular point Q of type $\frac{1}{5}(1, 3, 2)$, one point of type $\frac{1}{4}(1, 3, 1)$, and one point of type $\frac{1}{3}(1, 1, 2)$.

We have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\sigma_Q"'] \arrow[dr, "\sigma_P"] \arrow[rrr, "\eta"] & & & \mathbb{P}(1, 3, 4), \\
Y_P \arrow[dr, "\pi_P"'] & & Y_Q \arrow[dl, "\pi_Q"] & & \\
 & X \arrow[rrruu, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,3,4)$,
- π_Q is the Kawamata blow up at the point Q with weights $(1,3,2)$,
- σ_Q is the Kawamata blow up with weights $(1,3,2)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,3,4)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

We may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{P, Q\Big\}$$

due to Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7.

The exceptional divisor E_Q of the birational morphism π_Q contains two singular point Q_1 and Q_2 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$, respectively.

Lemma 2.7.1. *The set $\mathbb{CS}(Y_Q, \frac{1}{n}\mathcal{M}_{Y_Q})$ contain neither the point Q_1 nor the point Q_2.*

Proof. Suppose that the set $\mathbb{CS}(Y_Q, \frac{1}{n}\mathcal{M}_{Y_Q})$ contains the point Q_1. We then consider the Kawamata blow up $\alpha_1 : U_1 \to Y_Q$ at the point Q_1 with weights $(1,1,2)$. Let $\mathcal{D}_1$ be the proper transform of the linear system $|-7K_X|$ by the birational morphism $\alpha_1 \circ \pi_Q$. We have

$$\mathcal{M}_{U_1} \sim_{\mathbb{Q}} -nK_{U_1} \sim_{\mathbb{Q}} (\pi_Q \circ \alpha_1)^*(-nK_X) - \frac{n}{5}\alpha_1^*(E_Q) - \frac{n}{3}F_1,$$

where F_1 is the exceptional divisor of α_1. Let D_1 be a general surface in $\mathcal{D}_1$. Because the base locus of the linear system $\mathcal{D}_1$ does not contain any curve, the divisor D_1 is nef. We then see

$$D_1 \sim_{\mathbb{Q}} (\pi_Q \circ \alpha_1)^*(-7K_X) - \frac{2}{5}\alpha_1^*(E_Q) - \frac{2}{3}F_1,$$

which shows

$$\begin{aligned} D_1 \cdot M_1 \cdot M_2 &= \Big((\pi_Q \circ \alpha_1)^*(-7K_X) - \frac{2}{5}\alpha_1^*(E_Q) - \frac{2}{3}F_1\Big)\Big((\pi_Q \circ \alpha_1)^*(-nK_X) - \frac{n}{5}\alpha_1^*(E_Q) - \frac{n}{3}F_1\Big)^2 \\ &= -\frac{1}{12}n^2, \end{aligned}$$

where M_1 and M_2 are general surfaces in $\mathcal{M}_{U_1}$. It is a contradiction.

To exclude the point Q_2, we use the exactly same method. □

Meanwhile, the exceptional divisor E_P of the birational morphism π_P contains two singular point P_1 and P_2 that are quotient singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively.

Lemma 2.7.2. *If the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the point P_1, then $\mathcal{M} = |-3K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the point P_1. Let $\beta_1 : W_1 \to Y_P$ be the Kawamata blow up at the point P_1 with weights $(1,3,1)$. Also, let $\mathcal{L}_1$ be the proper transform of the linear system $|-3K_X|$ by the birational morphism $\beta_1 \circ \pi_P$. Then, the base locus of the linear system $\mathcal{L}_1$ consists of the irreducible curve C_{W_1}. Let H_1 be a general surface in $\mathcal{L}_1$. Then, we have

$$H_1 \sim_{\mathbb{Q}} (\pi_P \circ \beta_2)^*(-3K_X) - \frac{3}{7}\beta_2^*(E_P) - \frac{3}{4}G_1,$$

where G_1 is the exceptional divisor of β_1. Then, the inequality $-K_{W_1} \cdot C_{W_1} = -3K_{W_1}^3 < 0$ and the equivalence $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ imply $\mathcal{M} = |-3K_X|$ by Theorem 0.2.9. □

Lemma 2.7.3. *The set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ cannot contain the point P_2.*

Proof. Suppose that the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the point P_2. Let $\beta_2 : W_2 \to Y_P$ be the Kawamata blow up at the point P_2 with weights $(1,2,1)$ and $\mathcal{L}_2$ be the proper transform of the linear system $|-4K_X|$ by the birational morphism $\beta_2 \circ \pi_P$. Then, the base locus of the linear system $\mathcal{L}_2$ consists of the irreducible curve $\bar{C}_{W_2}$.

Let H_2 be a general surface in $\mathcal{L}_2$. Then, we have

$$H_2 \sim_{\mathbb{Q}} (\pi_P \circ \beta_2)^*(-4K_X) - \frac{4}{7}\beta_2^*(E_P) - \frac{1}{3}G_2,$$

where G_2 is the exceptional divisor of β_2. The equivalence $\mathcal{M}_{W_2} \sim_{\mathbb{Q}} -nK_{W_2}$ implies $M|_{H_2} \equiv n\bar{C}_{W_2}$, where M is a general surface in $\mathcal{M}_{W_2}$. Therefore, we obtain the identity $\mathcal{M} = |-4K_X|$ from Theorem 0.2.9 because $\bar{C}_{W_2}^2 = -\frac{1}{30}$ on the normal surface H_2. However, it is a contradiction because $|-4K_X|$ is not a pencil. $\square$

Consequently, we have proved

Proposition 2.7.4. *The linear system $|-3K_X|$ is the only Halphen pencil on X.*

In the case $\gimel = 61$, the hypersurface X is given by a general quasihomogeneous equation of degree 25 in $\mathbb{P}(1,4,5,7,9)$ with $-K_X^3 = \frac{5}{252}$. It has three singular points. One is a quotient singular point P of type $\frac{1}{9}(1,4,5)$, another is a quotient singular point Q of type $\frac{1}{7}(1,5,2)$, and the other is a quotient singular point of type $\frac{1}{4}(1,3,1)$.

We have a commutative diagram

$$\begin{tikzcd}
& V \arrow[dl, "\sigma_Q"'] \arrow[dr, "\sigma_P"] \arrow[rrrd, "\eta"] & & & \\
Y_P \arrow[dr, "\pi_P"'] & & Y_Q \arrow[dl, "\pi_Q"] & & \mathbb{P}(1,4,5), \\
& X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,4,5)$,
- π_Q is the Kawamata blow up at the point Q with weights $(1,5,2)$,
- σ_Q is the Kawamata blow up with weights $(1,5,2)$ at the point Q_1 whose image by the birational morphism π_P is the point Q,
- σ_P is the Kawamata blow up with weights $(1,4,5)$ at the point P_1 whose image by the birational morphism π_Q is the point P,
- η is an elliptic fibration.

Proposition 2.7.5. *The linear system $|-4K_X|$ is the only Halphen pencil on X.*

Proof. The proof is exactly same as that of the case $\gimel = 40$. $\square$

2.8. Case $\gimel = 43$, hypersurface of degree 20 in $\mathbb{P}(1,2,4,5,9)$.

The threefold X is a general hypersurface of degree 20 in $\mathbb{P}(1,2,4,5,9)$ with $-K_X^3 = \frac{1}{18}$. The singularities of the hypersurface X consist of five points that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and the point $O = (0:0:0:0:1)$ that is a quotient singularity of type $\frac{1}{9}(1,4,5)$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,4),
\end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,4,5)$,
- β is the Kawamata blow up with weights $(1,4,1)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,4,1)$ contained in the exceptional divisor of α,
- η is an elliptic fibration.

Due to Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, and Corollary 0.3.7, we may assume $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$.

Let $E \cong \mathbb{P}(1,4,5)$ be the exceptional divisor of the birational morphism α and D be a general surface of the pencil $|-2K_X|$. The linear system $|-2K_U|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the curve C_U and the unique curve L in the linear system $|\mathcal{O}_{\mathbb{P}(1,4,5)}(1)|$ on the surface E.

The exceptional divisor E contains two singular points P and Q of U that are quotient singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{5}(1,4,1)$, respectively.

Lemma 2.8.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain a curve.*

Proof. Suppose that it contains a curve Z. Then, $-K_U \cdot Z = \frac{1}{9}$ by Lemma 0.2.7. Because $-K_U^3 = \frac{1}{20}$, it contradicts Lemma 0.2.3. □

Therefore, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ must contain either the point P or the point Q.

Lemma 2.8.2. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point P, then $\mathcal{M} = |-2K_X|$*

Proof. Let $\gamma : W \to U$ be the Kawamata blow up at the point P with weights $(1,3,1)$. The exceptional divisor of γ contains the singular point P_1 of the variety W that is a quotient singularity of type $\frac{1}{3}(1,2,1)$. The pencil $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$.

We first suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_1. Let $\pi : V \to W$ be the Kawamata blow up at the singular point P_1 with weights $(1,2,1)$ and $\mathcal{H}$ be the proper transform, on the threefold V, of the linear system $|-5K_X|$. The pencil $|-2K_V|$ is the proper transform of the pencil $|-2K_X|$. The base locus of the linear system $\mathcal{H}$ consists of the irreducible curve $\tilde{C}_V$ whose image to X is the base locus of the linear system $|-5K_X|$. For a general surface H in $\mathcal{H}$, we have $H \cdot \tilde{C}_V = 1$ and $H^3 = 6$, which implies that the divisor H is nef and big. On the other hand, we have $H \cdot M_V \cdot D_V = 0$, where M_V is general surface in $\mathcal{M}_V$. Therefore, it follows from Theorem 0.2.9 that $\mathcal{M} = |-2K_X|$.

From now, we suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ does not contain the point P_1, which implies that the log pair $(W, \frac{1}{n}\mathcal{M}_W)$ is terminal by Lemma 0.2.7.

In a neighborhood of the point O, the monomials x, z, and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 4$, and $\mathrm{wt}(t) = 5$. Then, in a neighborhood of the singular point P, the Kawamata blow up α is given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{9}},\ z = \tilde{z}^{\frac{4}{9}},\ t = \tilde{t}\tilde{z}^{\frac{5}{9}},$$

where $\tilde{x}$, $\tilde{z}$, and $\tilde{t}$ are weighted local coordinated on the variety U in a neighborhood of the singular point P such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$, and $\mathrm{wt}(\tilde{t}) = 1$. The surface E is given by the equation $\tilde{z} = 0$, and the surface S_U is given by the equation $\tilde{x} = 0$. Moreover, it follows from the local equation of the surface D_U that $D_U \cdot S_U = C_U + 2L$, where the curve L is locally given by the equations $\tilde{z} = \tilde{x} = 0$.

In a neighborhood of the point P_1, the birational morphism γ is given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{4}},\ \tilde{z} = \bar{z}^{\frac{3}{4}},\ \tilde{t} = \bar{t}\bar{z}^{\frac{1}{4}},$$

where $\bar{x}$, $\bar{z}$, and $\bar{t}$ are weighted local coordinates on the variety W in a neighborhood of the point P_1 such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{z}) = 2$, and $\mathrm{wt}(\bar{t}) = 1$. In particular, the exceptional divisor of the birational morphism γ is given by the equation $\bar{z} = 0$ and the surface S_W is locally given by the equation $\bar{x} = 0$.

Let F be the exceptional divisor of the birational morphism γ and $\bar{L}$ be the curve on the variety W that is locally given by the equations $\bar{z}=\bar{x}=0$. Then,

$$D_W\cdot S_W=C_W+2L_W+\bar{L},\ D_W\cdot E_W=2L_W,\ D_W\cdot F=2\bar{L},$$

while the base locus of $|-2K_W|$ consists of the curves C_W, L_W, and $\bar{L}$. We have

$$D_W\cdot C_W=0,\ D_W\cdot L_W=-\frac{2}{5},\ D_W\cdot\bar{L}=\frac{2}{3},$$

because

$$\begin{cases} D_W\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\dfrac{2}{9}\gamma^*(E)-\dfrac{2}{4}F,\\ S_W\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-K_X)-\dfrac{1}{9}\gamma^*(E)-\dfrac{1}{4}F,\\ E_W\sim_{\mathbb{Q}}\gamma^*(E)-\dfrac{3}{4}F.\end{cases}$$

We also have

$$\begin{cases} S_W^y\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\dfrac{11}{9}\gamma^*(E)-\dfrac{6}{4}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-2K_X)-\dfrac{11}{9}E_W-\dfrac{5}{3}F,\\ S_W^t\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-5K_X)-\dfrac{5}{9}\gamma^*(E)-\dfrac{1}{4}F\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-5K_X)-\dfrac{5}{9}E_W-\dfrac{2}{3}F,\end{cases}$$

which implies that

$$-14K_W\sim 14D_W\sim 2S_W^y+2S_W^t+2E_W.$$

The support of the cycle $S_W^y\cdot S_W^t$ does not contain the curves $\bar{L}$ and C_W. Therefore, the base locus of the linear system $|-14K_W|$ does not contain curves except the curve L_W.

The log pair $(W,\lambda|-14K_W|)$ is log-terminal for some rational number $\lambda>\frac{1}{14}$ but the divisor $K_W+\lambda|-14K_W|$ has non-negative intersection with all curves on the variety W except the curve L_W. Hence, it follows from [21] that there is a log-flip $\zeta: W\dashrightarrow W'$ along the curve L_W with respect to the log pair $(W,\lambda|-14K_W|)$. In particular, the divisor $-K_{W'}$ is nef. Thus, the singularities of the log pair $(W',\frac{1}{n}\mathcal{M}_{W'})$ are terminal because the singularities of the log pair $(W,\frac{1}{n}\mathcal{M}_W)$ are terminal but the rational map ζ is a log flop with respect to the log pair $(W,\frac{1}{n}\mathcal{M}_W)$. Hence, the divisor $-K_{W'}$ is not big by Theorem 0.2.4, and hence the divisor $-K_W$ is not big either.

The rational functions $(\alpha\circ\gamma)^*(\frac{y}{x^2})$ and $(\alpha\circ\gamma)^*(\frac{ty}{x^7})$ are contained in the linear systems $|2S_W|$ and $|7S_W|$, respectively. The equivalences

$$S_W^z\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-4K_X)-\frac{4}{9}\gamma^*(E)\sim_{\mathbb{Q}}(\alpha\circ\gamma)^*(-4K_X)-\frac{4}{9}E_W-\frac{1}{3}F$$

imply that $-6K_W\sim S_W^y+S_W^z+E_W$. Thus, the rational function $(\alpha\circ\gamma)^*(\frac{zy}{x^6})$ is contained in the linear system $|6S_W|$, which implies that the linear system $|-42K_W|$ maps the variety W dominantly on a threefold[5]. Hence, the divisor $-K_W$ is big, which is a contradiction. □

Therefore, we may assume that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains the point Q. The exceptional divisor G of β contains the singular point Q_1 of Y that is a quotient singularity of type $\frac{1}{3}(1,2,1)$. However, we have the following statement.

Lemma 2.8.3. *If the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1, then $\mathcal{M}=|-2K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1. Let $\sigma_1: Y_1\to Y$ be the Kawamata blow up at the singular point Q_1. Then, the pencil $|-2K_{Y_1}|$ is the proper transform of the pencil $|-2K_X|$ and its base locus consists of the curves C_{Y_1} and L_{Y_1}. Thus, we see that

$$-K_{Y_1}\cdot L_{Y_1}=-\frac{1}{4},\quad -K_{Y_1}\cdot C_{Y_1}=0$$

[5]In fact, the linear system $|-210K_W|$ induces a birational map $W\dashrightarrow X'$, where X' is a hypersurface of degree 30 in $\mathbb{P}(1,2,6,7,15)$ with canonical singularities.

because $D_{Y_1} \cdot S_{Y_1} = C_{Y_1} + 2L_{Y_1}$. Hence, the divisor $B := D_{Y_1} + (\beta \circ \sigma_1)^*(-10K_U)$ is nef and big. On the other hand, $B \cdot M_{V_1} \cdot D_{V_1} = 0$, where M_{Y_1} is a general surface in $\mathcal{M}_{Y_1}$. Therefore, we obtain $\mathcal{M}_{Y_1} = |-2K_{Y_1}|$ from Theorem 0.2.9. $\square$

Hence, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain subvarieties of G. However, it is not empty by Theorem 0.2.4. Therefore, it must consist of the point $\bar{P}$ whose image to U is the point P.

Let $\sigma_2 : Y_2 \to Y$ be the Kawamata blow up of the singular point $\bar{P}$. The proof of Lemma 2.8.2 shows that the pencil $\mathcal{M}$ coincides with $|-2K_X|$ if the set $\mathbb{CS}(Y_2, \frac{1}{n}\mathcal{M}_{Y_2})$ contains the singular point of the variety Y_2 that is a quotient singularity of type $\frac{1}{3}(1,2,1)$ contained in the exceptional divisor of the birational morphism σ_2. Therefore, we may assume that the log pair $(Y_2, \frac{1}{n}\mathcal{M}_{Y_2})$ is terminal by Lemma 0.2.7.

The pencil $|-2K_{Y_2}|$ is the proper transform of the pencil $|-2K_X|$ and its base locus consists of the curves C_{Y_2}, L_{Y_2}, and a curve $\tilde{L}$ contained in the exceptional divisor of σ_2. Moreover, the curve L_{Y_2} is the only curve that has negative intersection with the divisor $-K_{Y_2}$ because $D_{Y_2} \sim_{\mathbb{Q}} -2K_{Y_2}$ and $-K_{Y_2} \cdot C_{Y_2} = 0$. Hence, it follows from [21] that there is a log-flip $\chi : Y_2 \dashrightarrow Y_2'$ along the curve L_{Y_2} with respect to the log pair $(Y_2, \lambda|-2K_{Y_2}|)$ for some rational number $\lambda > \frac{1}{2}$. In particular, the divisor $-K_{Y_2}$ is nef.

The rational map χ is a log flop with respect to the log pair $(Y_2, \frac{1}{n}\mathcal{M}_{Y_2})$. Thus, the singularities of the log pair $(Y_2, \frac{1}{n}\mathcal{M}_{Y_2})$ are terminal. Hence, the divisor $-K_{Y_2}$ is not big by Theorem 0.2.4. On the other hand, the abundance theorem ([14]) implies that the linear system $|-rK_{Y_2}|$ is base-point-free for $r \gg 0$. Moreover, the pull-backs of the rational functions $\frac{y}{x^2}$ and $\frac{zy}{x^5}$ are contained in the linear systems $|2S_{Y_2}|$ and $|6S_{Y_2}|$, respectively. Thus, the linear system $|-rK_{Y_2}|$ induces an elliptic fibration, which is impossible by Theorem 0.2.4.

Consequently, we have proved

Proposition 2.8.4. *The linear system $|-2K_X|$ is the only Halphen pencil on X.*

2.9. Cases $\gimel = 49$ and 64.

We first consider the case $\gimel = 49$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 21 in $\mathbb{P}(1,3,5,6,7)$ with $-K_X^3 = \frac{1}{30}$. Then, the singularities of X consist of one quotient singular point P of type $\frac{1}{6}(1,5,1)$, one quotient singular point Q of type $\frac{1}{5}(1,3,2)$, and three quotient singular points of type $\frac{1}{3}(1,2,1)$.

We have the following commutative diagram:

$$\begin{tikzcd} & Y_P \arrow[dl, "\pi_P"'] \arrow[dr, "\eta_P"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,5) \end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1,5,1)$,
- η_P is an elliptic fibration.

We may assume that X is given by a quasihomogeneous equation

$$z^3t + z^2 f_{11}(x,y,t,w) + z f_{16}(x,y,t,w) + f_{21}(x,y,t,w) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. We then see that there is another elliptic fibration as follows:

$$
\begin{tikzcd}
 & Y_Q \arrow[dl, "\pi_Q"'] \arrow[dr, "\eta_Q"] & \\
X \arrow[rr, dashed, "\xi"'] & & G \subset \mathbb{P}^3
\end{tikzcd}
$$

where

- ξ is the rational map given by the linear system spanned by $\{x^6, x^3y, y^2, t\}$
- π_Q is the Kawamata blow up at the point Q with weights $(1,3,2)$.
- G is the image of $\xi : X \dashrightarrow \mathbb{P}^3$ that is a quadratic cone in $\mathbb{P}^3$,
- η_Q is an elliptic fibration.

It follows from Corollary 0.3.7 and Lemmas 0.3.3, 0.3.10 that

$$\Big\{P, Q\Big\} \supseteq \mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \neq \varnothing.$$

The exceptional divisor E_P of the birational morphism π_P contains one singular point P_1 that is a quotient singularity of type $\frac{1}{5}(1,4,1)$.

Lemma 2.9.1. *If the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ contains the point P_1, then $\mathcal{M} = |-3K_X|$.*

Proof. We can assume that

$$f_{21}(x,y,t,w) = t^3y + t^2g_9(x,y,w) + tg_{15}(x,y,w) + g_{21}(x,y,w),$$

where g_i is a general quasihomogeneous polynomial of degree i. Then, locally at the singular point P, the monomials x, z, and w can be considered as weighted local coordinates on the threefold X with weights $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 5$, and $\mathrm{wt}(w) = 7$, which implies that locally at the singular point P_1, the birational morphism π is given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{6}},\ z = \tilde{z}^{\frac{5}{6}},\ w = \tilde{w}\tilde{z}^{\frac{1}{6}},$$

where $\tilde{x}$, $\tilde{z}$, and $\tilde{w}$ can be considered as weighted local coordinates in a neighborhood of the singular point P_1 with weights $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 4$, and $\mathrm{wt}(\tilde{w}) = 1$.

Let D be a general surface in the linear system $|-3K_X|$. Then, it is given by an equation

$$\lambda x^3 + \mu y = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Locally at the point P_1, the proper transform D_{Y_P} is given by

$$\lambda\tilde{x}^3 + \mu\Big(\epsilon_1\tilde{w}^3 + \epsilon_2\tilde{w}^2\tilde{x} + \epsilon_3\tilde{w}\tilde{x} + \text{higher terms}\Big) = 0,$$

which implies that $|-3K_{Y_P}|$ is the proper transform of $|-3K_X|$ and that it has no base curves on the surface E_P. The exceptional divisor E_P is defined by $\tilde{z} = 0$.

Let $\alpha : U \to Y_P$ be the Kawamata blow up at the point P_1 with weights $(1,4,1)$ and let F_P be its exceptional divisor. Around the singular point of F_P, the birational morphism α can be given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{5}},\ \tilde{z} = \bar{z}^{\frac{4}{5}},\ \tilde{w} = \bar{w}\bar{z}^{\frac{1}{5}},$$

where $\bar{x}$, $\bar{z}$, and $\bar{w}$ are weighted local coordinates on the variety U in a neighborhood of the singular point of the surface F_P with weights $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$, and $\mathrm{wt}(\tilde{w}) = 1$.

The proper transform D_U is given by an equation of the form

$$\lambda\bar{x}^3 + \mu\Big(\epsilon_1\bar{w}^3 + \epsilon_2\bar{w}^2\bar{x} + \epsilon_3\bar{w}\bar{x} + \text{higher terms}\Big) = 0,$$

which shows that $|-3K_U|$ is the proper transform of $|-3K_X|$ and that $|-3K_U|$ does not have base curves on F_P. Hence, the base locus of $|-3K_U|$ consists of the curve C_U whose image to X is defined by the equations $x = y = 0$.

Then, the inequality $-K_U \cdot C_U = -3K_U^3 < 0$ and the equivalence $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ imply that the pencil $\mathcal{M}_W$ coincides with the pencil $|-3K_U|$ by Theorem 0.2.9. $\square$

The exceptional divisor E_Q contains two quotient singular points Q_1 and Q_2 of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively.

Lemma 2.9.2. *If the set $\mathbb{CS}(Y_Q, \frac{1}{n}\mathcal{M}_{Y_Q})$ contains the point Q_1, then $\mathcal{M} = |-3K_X|$.*

Proof. The same method for the proof of Lemma 2.9.1 implies that $\mathcal{M} = |-3K_X|$. □

Lemma 2.9.3. *The set $\mathbb{CS}(Y_Q, \frac{1}{n}\mathcal{M}_{Y_Q})$ cannot contain the point Q_2.*

Proof. Let $\beta : W \to Y_Q$ be the Kawamata blow up at Q_2 with weights $(1,1,2)$. Also, let F_Q be the exceptional divisor of the birational morphism β. Let $\mathcal{R}$ be the linear system given by the equations

$$\lambda x^6 + \mu x^3 y + \nu t = 0,$$

where $(\lambda : \mu : \nu) \in \mathbb{P}^2$. The base locus of the linear system $\mathcal{R}$ consists of the curve $\tilde{C}$ given by the equations $x = t = 0$, in other words, we have $\tilde{C} = R \cdot S$, where R is a general surface of the linear system $\mathcal{R}$.

Around the point Q, the monomials x, y, and w can be considered as weighted local coordinates on X with weights $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, and $\mathrm{wt}(w) = 2$. Also, around the singular point Q_2, the birational morphism π_Q is given by the equations

$$x = \tilde{x}\tilde{y}^{\frac{1}{5}},\ y = \tilde{y}^{\frac{3}{5}},\ w = \tilde{w}\tilde{y}^{\frac{2}{5}},$$

where $\tilde{x}$, $\tilde{y}$, and $\tilde{w}$ are weighted local coordinates around the singular point Q_2 with weights $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{y}) = 1$, and $\mathrm{wt}(\tilde{w}) = 2$. The proper transform R_Y is given by an equation of the form

$$\lambda\tilde{x}^6 + \mu\tilde{x}^3 + \nu\Big(\delta_1\tilde{w}^3 + \delta_2\tilde{x}\tilde{w} + \delta_3\tilde{x}^2\tilde{w}^2 + \delta_4\tilde{y}\tilde{x}^2 + \delta_5\tilde{y}^3 + \text{higher terms}\Big) = 0,$$

where $\delta_i \in \mathbb{C}$, which tells us that the proper transform $\mathcal{R}_Y$ has no base curve on the exceptional divisor E_Q.

Locally at the unique singular point of F_Q, the birational morphism β can be expressed by

$$\tilde{x} = \bar{x}\bar{w}^{\frac{1}{3}},\ \tilde{y} = \bar{y}\bar{w}^{\frac{1}{3}},\ \tilde{w} = \bar{w}^{\frac{2}{3}},$$

where $\bar{x}$, $\bar{y}$, and $\bar{w}$ are local coordinates with weight 1. Then, the surface R_W is given by an equation of the form

$$\lambda\bar{x}^6\bar{w} + \mu\bar{x}^3 + \nu\Big(\delta_1\bar{w} + \delta_2\bar{x} + \delta_3\bar{x}^2\bar{w} + \delta_4\bar{y}\bar{x} + \delta_5\bar{y}^3 + \text{higher terms}\Big) = 0,$$

which implies that the proper transform $\mathcal{R}_W$ does not have base curves on the surface F_Q either. The surface R_W is normal. We see

$$\begin{cases} R_W \sim_{\mathbb{Q}} (\pi_Q \circ \beta)^*\Big(-6K_X\Big) - \dfrac{6}{5}\beta^*(E_Q) - F_Q, \\ S_W \sim_{\mathbb{Q}} (\pi_Q \circ \beta)^*\Big(-K_X\Big) - \dfrac{1}{5}\beta^*(E_Q) - \dfrac{1}{3}F_Q. \end{cases}$$

These equivalences show that $\tilde{C}_W^2 < 0$ on the normal surface R_W because $S_W \cdot R_W = \tilde{C}_W$.

Let M be a general surface of the pencil $\mathcal{M}_W$. Then,

$$M\Big|_{R_W} \equiv -nK_W\Big|_{R_W} \equiv nS_W\Big|_{R_W} \equiv n\tilde{C}_W$$

by Lemma 0.2.6, which implies that $\mathcal{M} = \mathcal{R}$ by Theorem 0.2.9. It is a contradiction because the linear system $\mathcal{R}$ is not a pencil. □

Proposition 2.9.4. *If $\daleth = 49$, then $\mathcal{M} = |-a_1K_X|$.*

Proof. By the previous lemmas, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \{P, Q\}.$$

Furthermore, we also assume that the set $\mathbb{CS}(Y_P, \frac{1}{n}\mathcal{M}_{Y_P})$ consists of the point O whose image to X is the point Q.

Let $\gamma : V \to Y_P$ be the Kawamata blow up at the point O. Then, the proof of Lemma 2.9.1 implies that $|-3K_V|$ is the proper transform of the pencil $|-3K_X|$ and the base locus of $|-3K_V|$ consists of the curve C_V whose image to X is the base curve of the pencil $|-3K_X|$. Then, we can easily check that $\mathcal{M} = |-3K_X|$. $\square$

We now consider the case $\gimel = 64$. Let X be the hypersurface given by a general quasihomogeneous equation of degree 26 in $\mathbb{P}(1, 2, 5, 6, 13)$ with $-K_X^3 = \frac{1}{30}$. Then, the singularities of X consist of one quotient singular point P of type $\frac{1}{6}(1, 5, 1)$, one quotient singular point Q of type $\frac{1}{5}(1, 2, 3)$, and four quotient singular points of type $\frac{1}{2}(1, 1, 1)$.

We see the following commutative diagram:

$$\begin{tikzcd} & Y_P \arrow[dl, "\pi_P"'] \arrow[dr, "\eta_P"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5) \end{tikzcd}$$

where

- ψ is the natural projection,
- π_P is the Kawamata blow up at the point P with weights $(1, 5, 1)$,
- η_P is an elliptic fibration.

One the other hand, we may assume that X is given by a quasihomogeneous equation

$$z^4 t + z^3 f_{11}(x, y, t, w) + z^2 f_{16}(x, y, t, w) + z f_{21}(x, y, t, w) + f_{26}(x, y, t, w) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Therefore, there is another elliptic fibration as follows:

$$\begin{tikzcd} & Y_Q \arrow[dl, "\pi_Q"'] \arrow[dr, "\eta_Q"] & \\ X \arrow[rr, dashed, "\xi"'] & & G \subset \mathbb{P}^4 \end{tikzcd}$$

where

- ξ is the map given by the linear system spanned by $\{x^6, x^4 y, x^2 y^2, y^3, t\}$
- π_Q is the Kawamata blow up at the point Q with weights $(1, 2, 3)$.
- G is the image of $\xi : X \dashrightarrow \mathbb{P}^3$ that is isomorphic to $\mathbb{P}(1, 1, 3)$,
- η_Q is an elliptic fibration.

Proposition 2.9.5. *If* $\gimel = 64$*, then* $\mathcal{M} = |-a_1 K_X|$.

Proof. The proof is the same as that of Proposition 2.9.4. $\square$

2.10. Case $\gimel = 56$, hypersurface of degree 24 in $\mathbb{P}(1, 2, 3, 8, 11)$.

The threefold X is a general hypersurface of degree 24 in $\mathbb{P}(1, 2, 3, 8, 11)$ with $-K_X^3 = \frac{1}{22}$. Its singularities consist of three points that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and the point $O = (0 : 0 : 0 : 0 : 1)$ that is a quotient singularity of type $\frac{1}{11}(1, 3, 8)$.

Before we proceed, let us first describe some birational transformations of the hypersurface X with elliptic fibrations, which are useful to explain the geometrical nature of our proof. There

is a commutative diagram

$$
\begin{tikzcd}
& & Z \arrow[r, dashed, "\zeta"] \arrow[dl, "\nu"'] & Z' \arrow[r, "\sigma"] & \bar{Z}' \arrow[r, hook] & \mathbb{P}(1,2,5,14,21) \arrow[ddl, dashed, "\phi"] \\
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] & V \arrow[l, "\xi"'] \arrow[ul, "\omega"'] \arrow[r, dashed, "\chi"] & V' \arrow[d, "\upsilon"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3) \arrow[rr, dashed, "\rho"'] & & \mathbb{P}(1,2,5) &
\end{tikzcd}
$$

where

- ψ and ϕ are natural projections,
- α is the Kawamata blow up at the point O with weights $(1,3,8)$,
- β is the Kawamata blow up with weights $(1,3,5)$ at the singular point Q contained in the exceptional divisor E of α that is a quotient singularity of type $\frac{1}{8}(1,3,5)$,
- γ is the Kawamata blow up with weights $(1,3,2)$ at the singular point Q_1 of contained in the exceptional divisor F of β that is a quotient singularity of type $\frac{1}{5}(1,3,2)$,
- ν is the Kawamata blow up with weights $(1,1,2)$ at the singular point Q_2 of contained in the exceptional divisor of β that is a quotient singularity of type $\frac{1}{3}(1,1,2)$,
- ξ is the Kawamata blow up with weights $(1,1,2)$ at the point $\bar{Q}_2$ whose image to W is the point Q_2,
- ω is the Kawamata blow up with weights $(1,3,2)$ at the point $\bar{Q}_1$ whose image to W is the point Q_1,
- η and υ are elliptic fibrations,
- the maps ζ and χ are compositions of antiflips,
- the birational morphism σ is given by the plurianticanonical linear system of Z',
- the rational map ρ is a toric map,

The exceptional divisor E of the birational morphism α contains two singular points P and Q of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,5)$, respectively. Meanwhile, the exceptional divisor F of the birational morphism β also contains two singular points Q_1 and Q_2 of W that are quotient singularities of types $\frac{1}{5}(1,3,2)$ and $\frac{1}{3}(1,1,2)$, respectively.

Remark 2.10.1. The divisors $-K_{Z'}$, $-K_U$, and $-K_W$ are nef and big. Thus, the anticanonical models of the threefolds Z', U and W are Fano threefolds with canonical singularities. The anticanonical model of Z' is a hypersurface $\bar{Z}'$ of degree 42 in $\mathbb{P}(1,2,5,14,21)$. The anticanonical model of U is a hypersurface of degree 26 in $\mathbb{P}(1,2,3,8,13)$ and the anticanonical model of W is a hypersurface of degree 30 in $\mathbb{P}(1,2,3,10,15)$.

For the convenience, we denote the pencil $|-2K_X|$ by $\mathcal{B}$. In addition, a general surface in $\mathcal{B}$ is denoted by B and a general surface in $\mathcal{M}$ by M

It follows from Corollary 0.3.7 and Lemmas 0.3.3, 0.3.10 that we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$.

Lemma 2.10.2. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P, then $\mathcal{M} = \mathcal{B}$.*

Proof. Let $\beta_P : U_P \to U$ be the Kawamata blow up at the point P and E_P be its exceptional divisor. For a general surface D in $|-8K_X|$, we have

$$D_{U_P} \sim_{\mathbb{Q}} (\alpha \circ \beta_P)^*(-8K_X) - \frac{8}{11}\beta_P^*(E) - \frac{2}{3}E_P.$$

Because the base locus of the proper transform of the linear system $|-8K_X|$ on U_P does not contain any curve, the divisor D_{U_P} is nef and big.

Since $M_{U_P} \sim_{\mathbb{Q}} nS_{U_P}$ by Lemma 0.2.6 and $B_{U_P} \sim_{\mathbb{Q}} 2S_{U_P}$, we obtain

$$D_{U_P} \cdot B_{U_P} \cdot S_{U_P} = 2n\Big(\beta_P^*(-8K_U) - \frac{2}{3}E_P\Big) \cdot \Big(\beta_P^*(-K_U) - \frac{1}{3}E_P\Big)^2 = 0.$$

It implies $\mathcal{M}=\mathcal{B}$ by Theorem 0.2.9. □

Due to Theorem 0.2.4 and Lemma 0.2.7, we may assume that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ consists of the singular point Q. Thus, it follows from Theorem 0.2.4, Lemmas 0.2.3, 0.2.7 that

$$\varnothing\neq\mathbb{CS}\Big(W,\frac{1}{n}\mathcal{M}_W\Big)\subseteq\Big\{Q_1,\ Q_2\Big\}.$$

Now, we consider some local computation. We may assume that X is given by the equation

$$w^2y+wf_{13}(x,y,z,t)+f_{24}(x,y,z,t)=0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. The surface B is given by the equation $\lambda x^2+\mu y=0$, where $(\lambda:\mu)\in\mathbb{P}^1$. The base locus of $\mathcal{B}$ consists of the irreducible curve C that is given by $x=y=0$. We have $B\cdot S=C$.

In a neighborhood of O, the monomials x, z, and t can be considered as weighted local coordinates on X such that $\mathrm{wt}(x)=1$, $\mathrm{wt}(z)=3$, and $\mathrm{wt}(z)=8$. Then, in a neighborhood of the singular point O, the surface B can be given by equation

$$\lambda x^2+\mu\Big(\epsilon_1x^{13}+\epsilon_2zx^{10}+\epsilon_3z^2x^7+\epsilon_4z^3x^4+\epsilon_5z^4x+\epsilon_6tx^5+\epsilon_7tzx^2+\epsilon_8t^3+\epsilon_9z^8+\text{other terms}\Big)=0,$$

where $\epsilon_i\in\mathbb{C}$. In a neighborhood of the singular point Q, the birational morphism α can be given by the equations

$$x=\bar{x}\bar{t}^{\frac{1}{11}},\ z=\bar{z}\bar{t}^{\frac{3}{11}},\ t=\bar{t}^{\frac{8}{11}},$$

where $\bar{x}$, $\bar{z}$, and $\bar{t}$ are weighted local coordinates on U in a neighborhood of the singular point Q such that $\mathrm{wt}(\bar{x})=1$, $\mathrm{wt}(\bar{z})=3$, and $\mathrm{wt}(\bar{t})=8$. Thus, in a neighborhood of the singular point Q, the divisor E is given by the equation $\bar{t}=0$, the divisor S_U is given by $\bar{x}=0$, and the divisor B_U is given by the equation

$$\lambda\bar{x}^2+\mu\Big(\epsilon_1\bar{x}^{13}\bar{t}+\cdots+\epsilon_5\bar{z}^4\bar{x}\bar{t}+\epsilon_6\bar{t}\bar{x}^5+\epsilon_7\bar{t}\bar{z}\bar{x}^2+\epsilon_8\bar{t}^2+\epsilon_9\bar{z}^8\bar{t}^2+\text{other terms}\Big)=0,$$

which implies that $B_U\sim_{\mathbb{Q}}2S_U$ and the base locus of $\mathcal{B}_U$ is the union of C_U and the curve $L\subset E$ that is given by $\bar{x}=\bar{t}=0$. We have $E\cong\mathbb{P}(1,3,8)$ and the curve L is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,3,8)}(1)|$ on the surface E. The surface B_U is not normal. Indeed, B_U is singular at a generic point of L. We have $S_U\cdot B_U=C_U+2L$ and $E\cdot B_U=2L$, which implies that $S_U\cdot C_U=0$ and $S_U\cdot L=\frac{1}{24}$.

Lemma 2.10.3. *If the set $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)$ consists of the point Q_2, then $\mathcal{M}=\mathcal{B}$.*

Proof. In a neighborhood of Q_2, the birational morphism β can be given by the equations

$$\bar{x}=\tilde{x}\tilde{z}^{\frac{1}{8}},\ \bar{z}=\tilde{z}^{\frac{3}{8}},\ \bar{t}=\tilde{t}\tilde{z}^{\frac{5}{8}},$$

where $\tilde{x}$, $\tilde{z}$, and $\tilde{t}$ are weighted local coordinates on W in a neighborhood of Q_2 such that $\mathrm{wt}(\tilde{x})=1$, $\mathrm{wt}(\tilde{z})=1$, and $\mathrm{wt}(\tilde{t})=2$. Thus, in a neighborhood of the singular point Q_2, the divisor F is given by the equation $\tilde{z}=0$, the divisor S_W is given by $\tilde{x}=0$, the divisor E_W is given by $\tilde{t}=0$, and the divisor B_W is given by the equation

$$\lambda\tilde{x}^2+\mu\Big(\epsilon_7\tilde{t}\tilde{z}\tilde{x}^2+\epsilon_8\tilde{t}^2\tilde{z}+\epsilon_9\tilde{z}^4\tilde{t}^2+\text{other terms}\Big)=0,$$

which implies that $B_W\sim_{\mathbb{Q}}2S_W$, the base locus of $\mathcal{B}_W$ is the union of C_W, L_W, and the curve $L'\subset F$ that is given by $\tilde{x}=\tilde{z}=0$.

The surface F is isomorphic to $\mathbb{P}(1,3,5)$ and the curve L' is the unique curve of the linear system $|\mathcal{O}_{\mathcal{P}(1,3,5)}(1)|$ on the surface F. The surface B_W is smooth at a generic point of L'. We have

$$S_W\cdot B_W=C_W+2L_W+L',\quad E_W\cdot B_W=2L_W,\quad F\cdot B_W=2L',$$

which implies that

$$S_W\cdot C_W=0,\quad S_W\cdot L_W=0,\quad S_W\cdot L'=\frac{1}{15}$$

because

$$\begin{cases} S_W \sim_{\mathbb{Q}} (\alpha\circ\beta)^*(-K_X) - \dfrac{1}{11}\beta^*(E) - \dfrac{1}{8}F, \\ B_W \sim_{\mathbb{Q}} (\alpha\circ\beta)^*(-2K_X) - \dfrac{2}{11}\beta^*(E) - \dfrac{2}{8}F, \\ E_W \sim_{\mathbb{Q}} \beta^*(E) - \dfrac{5}{8}F. \end{cases}$$

Let R be the exceptional divisor of ν. Let O_2 be the singular point of Z that is contained in R. Then, $R \cong \mathbb{P}(1,1,2)$ and O_2 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on the threefold Z. In a neighborhood of O_2, the birational morphism ν can be given by the equations

$$\tilde{x} = \hat{x}\hat{t}^{\frac{1}{3}},\ \tilde{z} = \hat{z}\hat{t}^{\frac{1}{3}},\ \tilde{t} = \hat{t}^{\frac{2}{3}},$$

where $\hat{x}$, $\hat{z}$ and $\hat{t}$ are weighted local coordinates on Z in a neighborhood of O_2 with weight 1. Thus, in a neighborhood of the singular point O_2, the divisor R is given by the equation $\hat{t} = 0$, the divisor S_Z is given by $\hat{x} = 0$, the divisor E_Z does not pass through the point O_2, the divisor F_Z is given by $\bar{z} = 0$, and the divisor B_Z is given by the equation

$$\lambda\hat{x}^2 + \mu\Big(\epsilon_8\hat{t}\hat{z} + \epsilon_9\hat{z}^4\hat{t}^2 + \text{other terms}\Big) = 0,$$

which implies that $B_Z \sim_{\mathbb{Q}} 2S_Z$, the base locus of $\mathcal{B}_Z$ consists of C_Z, L_Z, L'_Z, and the curve L'' that is given by the equations $\hat{x} = \hat{t} = 0$. The curve L'' is the unique curve in $|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$ on the surface R. The surface B_Z is smooth at a generic point of L''. Therefore, we obtain

$$S_Z\cdot B_Z = C_Z + 2L_Z + L'_Z + L'',\ E_Z\cdot B_Z = 2L_Z,\ F_Z\cdot B_Z = 2L'_Z,\ R\cdot B_Z = 2L'',$$

which gives

$$S_Z\cdot C_Z = 0,\quad S_Z\cdot L_Z = -\frac{1}{3},\quad S_Z\cdot L'_Z = -\frac{1}{10},\quad S_Z\cdot L'' = \frac{1}{2}$$

because

$$\begin{cases} S_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X) - \dfrac{1}{11}(\beta\circ\nu)^*(E) - \dfrac{1}{8}\nu^*(F) - \dfrac{1}{3}R, \\ B_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X) - \dfrac{2}{11}(\beta\circ\nu)^*(E) - \dfrac{2}{8}\nu^*(F) - \dfrac{2}{3}R, \\ E_Z \sim_{\mathbb{Q}} (\beta\circ\nu)^*(E) - \dfrac{5}{8}\nu^*(F) - \dfrac{2}{3}R, \\ F_Z \sim_{\mathbb{Q}} \nu^*(F) - \dfrac{1}{3}R. \end{cases}$$

In particular, the curves L_Z and L'_Z are the only curves on the variety Z that have negative intersection with the divisor $-K_Z$.

Due to Lemma 0.2.7, either the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ contains the point O_2 or the log pair $(Z, \frac{1}{n}\mathcal{M}_Z)$ is terminal.

We first suppose that the log pair $(Z, \frac{1}{n}\mathcal{M}_Z)$ is not terminal. Then, the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ must contain the point O_2. Let $\pi_2 : Z_2 \to Z$ be the Kawamata blow up at the point O_2 and H be the exceptional divisor of π_2. Then, our local calculations imply that $B_{Z_2} \sim_{\mathbb{Q}} 2S_{Z_2}$ and the base locus of $\mathcal{B}_{Z_2}$ consists of the curves C_{Z_2}, L_{Z_2}, L'_{Z_2}, and L''_{Z_2}. Furthermore, we have

$$S_{Z_2}\cdot B_T = C_{Z_2} + 2L_{Z_2} + L'_{Z_2} + L''_{Z_2},\quad E_{Z_2}\cdot B_{Z_2} = 2L_{Z_2},$$

$$F_{Z_2}\cdot B_{Z_2} = 2L'_{Z_2},\quad R_{Z_2}\cdot B_{Z_2} = 2L''_{Z_2},$$

which implies that

$$S_{Z_2}\cdot C_{Z_2} = 0,\quad S_{Z_2}\cdot L_{Z_2} = -\frac{1}{3},\quad S_{Z_2}\cdot L'_{Z_2} = -\frac{3}{5},\quad S_{Z_2}\cdot L''_{Z_2} = 0,$$

because

$$\begin{cases} S_{Z_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu\circ\pi_2)^*(-K_X) - \dfrac{1}{11}(\beta\circ\nu\circ\pi_2)^*(E) - \dfrac{1}{8}(\nu\circ\pi_2)^*(F) - \dfrac{1}{3}\pi_2^*R - \dfrac{1}{2}H, \\ E_{Z_2} \sim_{\mathbb{Q}} (\beta\circ\nu\circ\pi_2)^*(E) - \dfrac{5}{8}(\nu\circ\pi_2)^*(F) - \dfrac{2}{3}\pi_2^*R, \\ F_{Z_2} \sim_{\mathbb{Q}} (\nu\circ\pi_2)^*(F) - \dfrac{1}{3}\pi_2^*R - \dfrac{1}{2}H, \\ R_{Z_2} \sim_{\mathbb{Q}} \pi_2^*R - \dfrac{1}{2}H, \end{cases}$$

The curves L_{Z_2} and L'_{Z_2} are the only curves on the variety Z_2 that have negative intersection with the divisor $-K_{Z_2}$. Moreover, we see

$$\Big(B_{Z_2} + (\beta\circ\nu\circ\pi_2)^*(-16K_U) + (\nu\circ\pi_2)^*(-18K_W)\Big)\cdot L_{Z_2} = 0,$$
$$\Big(B_{Z_2} + (\beta\circ\nu\circ\pi_2)^*(-16K_U) + (\nu\circ\pi_2)^*(-18K_W)\Big)\cdot L'_{Z_2} = 0,$$

and hence the divisor $D_{Z_2} := B_{Z_2} + (\beta\circ\nu\circ\pi_2)^*(-16K_U) + (\nu\circ\pi_2)^*(-18K_W)$ is nef and big because $-K_U$ and $-K_W$ are nef and big. Therefore, we obtain

$$D_{Z_2}\cdot B_{Z_2}\cdot M_{Z_2} = 0,$$

and hence $\mathcal{M} = \mathcal{B}$ by Theorem 0.2.9.

For now, we suppose that the log pair $(Z, \frac{1}{n}\mathcal{M}_Z)$ is terminal. We will derive a contradiction from this assumption, so that the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ must contain the point O_2.

The log pair $(Z, \epsilon B_Z)$ is terminal for some rational number $\epsilon > \frac{1}{2}$ but the divisor $K_Z + \epsilon B_Z$ has nonnegative intersection with all curves on the variety Z except the curves L_Z and L'_Z. It follows from [21] that there is a composition of antiflips $\zeta : Z \dashrightarrow Z'$ and the divisor $-K_{Z'}$ is nef. Then, the singularities of the log pair $(Z', \frac{1}{n}\mathcal{M}_{Z'})$ are terminal because the singularities of the log pair $(Z, \frac{1}{n}\mathcal{M}_Z)$ are terminal and the rational map ζ is a log flop with respect to the log pair $(Z, \frac{1}{n}\mathcal{M}_Z)$.

We obtain

$$\begin{cases} S_Z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X) - \dfrac{1}{11}(\beta\circ\nu)^*(E) - \dfrac{1}{8}\nu^*(F) - \dfrac{1}{3}R \\ \qquad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-K_X) - \dfrac{1}{11}E_Z - \dfrac{2}{11}F_Z - \dfrac{5}{11}R, \\ S_Z^y \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X) - \dfrac{13}{11}(\beta\circ\nu)^*(E) - \dfrac{5}{8}\nu^*(F) - \dfrac{2}{3}R \\ \qquad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-2K_X) - \dfrac{13}{11}E_Z - \dfrac{15}{11}F_Z - \dfrac{21}{11}R, \\ S_Z^z \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-3K_X) - \dfrac{3}{11}(\beta\circ\nu)^*(E) - \dfrac{3}{8}\nu^*(F) - \dfrac{1}{3}R \\ \qquad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-3K_X) - \dfrac{3}{11}E_Z - \dfrac{6}{11}F_Z - \dfrac{4}{11}R, \\ S_Z^t \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-8K_X) - \dfrac{8}{11}(\beta\circ\nu)^*(E) \\ \qquad \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\nu)^*(-8K_X) - \dfrac{8}{11}E_Z - \dfrac{5}{11}F_Z - \dfrac{7}{11}R. \end{cases}$$

from

$$F_Z \sim_{\mathbb{Q}} \nu^*(F) - \frac{1}{3}R, \quad E_Z \sim_{\mathbb{Q}} (\beta\circ\nu)^*(E) - \frac{5}{8}\nu^*(F) - \frac{2}{3}R.$$

Thus, the pull-backs of the rational functions $\frac{y}{x^2}$, $\frac{zy}{x^5}$ and $\frac{ty^3}{x^{14}}$ are contained in the linear systems $|2S_Z|$, $|5S_Z|$ and $|14S_Z|$, respectively. In particular, the complete linear system $|-70K_Z|$ induces a dominant rational map $Z \dashrightarrow \mathbb{P}(1, 2, 5, 14)$. Thus, the anticanonical divisor $-K_{Z'}$ is nef and big. It contradicts Theorem 0.2.4 because the log pair $(Z', \frac{1}{n}\mathcal{M}_{Z'})$ is terminal. $\square$

Due to the lemma above, we may assume that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. In particular, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ is not empty and each member of the linear system $\mathcal{M}_Y$ is contracted to a curve by the morphism η.

Let G be the exceptional divisor of γ. Then, G contains two singular points Q_1' and Q_2' of Y that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. Then,

$$\mathbb{CS}\Big(Y, \frac{1}{n}\mathcal{M}_Y\Big) \subseteq \Big\{Q_1',\ Q_2',\ \bar{Q}_2\Big\},$$

where $\bar{Q}_2$ is the point on Y whose image to W by γ is the point Q_2.

In a neighborhood of Q_1, the birational morphism β can be given by the equations

$$\bar{x} = \tilde{x}\tilde{z}^{\frac{1}{8}},\ \bar{z} = \tilde{z}\tilde{t}^{\frac{3}{8}},\ \bar{t} = \tilde{t}^{\frac{5}{8}},$$

where $\tilde{x}$, $\tilde{z}$, and $\tilde{t}$ are weighted local coordinates on W in a neighborhood of Q_1 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$, and $\mathrm{wt}(\tilde{t}) = 2$. Thus, in a neighborhood of the singular point Q_1, the divisor F is given by the equation $\tilde{t} = 0$, the divisor S_W is given by $\tilde{x} = 0$, the divisor E_W does not pass though the point Q_1, and the divisor B_W is given by the equation

$$\lambda\tilde{x}^2 + \mu\Big(\epsilon_8\tilde{t} + \epsilon_9\tilde{z}^8\tilde{t}^4 + \text{other terms}\Big) = 0.$$

Therefore, $B_W \sim_{\mathbb{Q}} 2S_W$ and the base locus of $\mathcal{B}_W$ is the union of C_W, L_W and the curve L' that is given by the equations $\tilde{x} = \tilde{t} = 0$. We have

$$S_W \cdot B_W = C_W + 2L_W + L',\ E_W \cdot B_W = 2L_W,\ F \cdot B_W = 2L',$$

which gives us

$$S_W \cdot C_W = S_W \cdot L_U = 0, \quad S_W \cdot L' = \frac{1}{15}.$$

In a neighborhood of Q_2, the birational morphism γ can be given by the equations

$$\tilde{x} = \hat{x}\hat{t}^{\frac{1}{5}},\ \tilde{z} = \hat{z}^{\frac{3}{5}},\ \tilde{t} = \hat{t}\hat{z}^{\frac{2}{5}},$$

where $\hat{x}$, $\hat{z}$, and $\hat{t}$ are weighted local coordinates on Y in the neighborhood of Q_2' such that $\mathrm{wt}(\hat{x}) = 1$, $\mathrm{wt}(\hat{z}) = 1$, and $\mathrm{wt}(\hat{t}) = 2$. Thus, in a neighborhood of the singular point Q_2', the divisor G is given by the equation $\hat{z} = 0$, the divisor S_Y is given by $\hat{x} = 0$, the divisor F_Y is given by the equation $\bar{t} = 0$, and the divisor B_Y is given by the equation

$$\lambda\hat{x}^2 + \mu\Big(\epsilon_8\hat{t} + \epsilon_9\hat{z}^6\hat{t}^2 + \text{other terms}\Big) = 0.$$

Thus, $B_Y \sim_{\mathbb{Q}} 2S_Y$ and that the base locus of $\mathcal{B}_Y$ is the union of the irreducible curves C_Y, L_Y, and L_Y'. We have

$$\begin{cases} S_Y \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma)^*(-K_X) - \dfrac{1}{11}(\beta\circ\gamma)^*(E) - \dfrac{1}{8}\gamma^*(F) - \dfrac{1}{5}G, \\ E_Y \sim_{\mathbb{Q}} (\beta\circ\gamma)^*(E) - \dfrac{5}{8}\gamma^*(F), \\ F_Y \sim_{\mathbb{Q}} \gamma^*(F) - \dfrac{2}{5}G, \end{cases}$$

and

$$S_Y \cdot C_Y = S_Y \cdot L_Y = S_Y \cdot L_Y' = 0,$$

which simply means that C_Y, L_Y and L_Y' are components of a fiber of η.

Lemma 2.10.4. *If the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains Q_2', then $\mathcal{M} = \mathcal{B}$.*

Proof. Let $\sigma_2 : Y_2 \to Y$ be the Kawamata blow up at the point Q_2' and let H_2 be the exceptional divisor of σ_2. Then, our local calculations imply that $B_{Y_2} \sim_{\mathbb{Q}} 2S_{Y_2}$ and the base locus of $\mathcal{B}_{Y_2}$ is the union of curves C_{Y_2}, L_{Y_2}, and L'_{Y_2}. Thus, we have

$$S_{Y_2} \cdot B_{Y_2} = C_{Y_2} + 2L_{Y_2} + L'_{Y_2}, \quad E_{Y_2} \cdot B_{Y_2} = 2L_{Y_2}, \quad F_{Y_2} \cdot B_{Y_2} = 2L'_{Y_2},$$

which implies that

$$S_{Y_2} \cdot C_{Y_2} = 0, \quad S_{Y_2} \cdot L_{Y_2} = 0, \quad S_{Y_2} \cdot L'_{Y_2} = -\frac{1}{3}$$

because

$$\begin{cases} S_{Y_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma\circ\sigma_2)^*(-K_X) - \frac{1}{11}(\beta\circ\gamma\circ\sigma_2)^*(E) - \frac{1}{8}(\gamma\circ\sigma_2)^*(F) - \frac{1}{5}\sigma_2^*(G) - \frac{1}{3}H_2, \\ E_{Y_2} \sim_{\mathbb{Q}} (\beta\circ\gamma\circ\sigma_2)^*(E) - \frac{5}{8}(\gamma\circ\sigma_2)^*(F), \\ F_{Y_2} \sim_{\mathbb{Q}} (\gamma\circ\sigma_2)^*(F) - \frac{2}{5}\sigma_2^*(G) - \frac{2}{3}H_2, \\ G_{Y_2} \sim_{\mathbb{Q}} \sigma_2^*(G) - \frac{1}{3}H_2, \end{cases}$$

The curve L'_{Y_2} is the only curve on Y_2 that has negative intersection with $-K_{Y_2}$. Moreover, we have $(S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W)) \cdot L'_{Y_2} = 0$, which implies that the divisor $S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W)$ is nef and big because $-K_W$ is nef and big. Therefore,

$$(S_{Y_2} + (\gamma\circ\sigma_2)^*(-5K_W)) \cdot B_{Y_2} \cdot M_{Y_2} = 0$$

by Lemma 0.2.6, and hence $\mathcal{M} = \mathcal{B}$ by Theorem 0.2.9 □

Lemma 2.10.5. *The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ cannot contain the point Q_1'.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1'. Let $\sigma_1 : Y_1 \to Y$ be the Kawamata blow up at the point Q_1'.

Let $\mathcal{D}$ be a general pencil in the linear system $|-3K_X|$. Then, the base curve of $\mathcal{D}$ is the curve $\bar{C}$ given by $x = z = 0$. Moreover, the base locus of $\mathcal{D}_{Y_1}$ consists of the curve $\bar{C}_{Y_1}$. Thus, we see that $\bar{C}_{Y_1} = S_{Y_1} \cdot D_{Y_1}$ for a general surface D_{Y_1} in $\mathcal{D}_{Y_1}$. On the other hand, we have $D_{Y_1} \cdot C_{Y_1} < 0$, which implies that $n = 3$ and $\mathcal{M}_{Y_1} = \mathcal{D}_{Y_1}$ by Theorem 0.2.9. However, $D_{Y_1} \not\sim_{\mathbb{Q}} -3K_{Y_1}$. □

Consequently, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ consists of the point $\bar{Q}_2$ whose image to W is the point Q_2. It implies that $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W) = \{Q_1, Q_2\}$. We have $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$ by Lemma 0.2.6.

Let H be the exceptional divisor of ξ. Then, it follows from the local computations made during the proof of Lemma 2.10.3 that $B_V \sim_{\mathbb{Q}} 2S_V$. The base locus of $\mathcal{B}_V$ is the union of the irreducible curves C_V, L_V, L'_V, and the curve L'' such that $L'' = S_V \cdot H$. We have

$$S_V \cdot C_V = 0, \quad S_V \cdot L_V = -\frac{1}{3}, \quad S_V \cdot L'_V = -\frac{1}{6}, \quad S_V \cdot L'' = \frac{1}{2}.$$

Let $\bar{O}$ be the singular point of V that is contained in H. Then, $\omega(\bar{O})$ is the singular point of Z contained in the exceptional divisor of ν. It follows from Lemma 0.2.7 that either the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point $\bar{O}$ or the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ is terminal.

Suppose that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point $\bar{O}$. Then, the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ contains the point $\omega(\bar{O})$. The proof of Lemma 2.10.3 shows that $\mathcal{M} = \mathcal{B}$ if the set $\mathbb{CS}(Z, \frac{1}{n}\mathcal{M}_Z)$ contains the point $\omega(\bar{O}) = O_2$.

From now, we suppose that the singularities of the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ are terminal. The singularities of the log pair $(V, \epsilon B_V)$ are log-terminal for some rational number $\epsilon > \frac{1}{2}$ but the divisor $K_V + \epsilon B_V$ has nonnegative intersection with all curves on the variety V except the curves L_V and L'_V. Then, there is a composition of antiflips $\chi : V \dashrightarrow V'$ and the divisor $-K_{V'}$ is nef.

Hence, the linear system $|-rK_{V'}|$ is base-point-free for $r \gg 0$ by the log abundance theorem ([14]).

It follows from the proof of Lemma 2.10.3 that the pull-backs of the rational functions $\frac{y}{x^2}$ and $\frac{zy}{x^5}$ are contained in the linear systems $|2S_V|$ and $|5S_V|$, respectively. In particular, the complete linear system $|-10K_V|$ induces a dominant rational map $V \dashrightarrow \mathbb{P}(1,2,5)$, which implies that the linear system $|-rK_{V'}|$ induces a dominant morphism to a surface. In fact, the linear system $|-rK_{V'}|$ induces the morphism υ. The singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ are terminal because the singularities of the log pair $(V, \frac{1}{n}\mathcal{M}_V)$ are terminal and the rational map χ is a log flop with respect to the log pair $(V, \frac{1}{n}\mathcal{M}_V)$. However, the singularities of the log pair $(V', \frac{1}{n}\mathcal{M}_{V'})$ cannot be terminal by Theorem 0.2.4. We have obtained a contradiction.

Summing up, we have proved

Proposition 2.10.6. *The linear system $|-2K_X|$ is a unique Halphen pencil on X.*

2.11. Cases $\gimel = 63,\ 77,\ 83,$ and 85.

Suppose that $\gimel \in \{63, 83\}$. Then, the threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ always contains the point $O = (0:0:0:1:0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, a_2, a_3 - a_2)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2), \end{tikzcd}$$

where

- α is the Kawamata blow up at the point O with weights $(1, a_2, a_3 - a_2)$,
- β is the Kawamata blow up with weights $(1, a_2, a_3 - 2a_2)$ at the point P of U that is a quotient singularity of type $\frac{1}{a_3 - a_2}(1, a_2, a_3 - 2a_2)$,
- η is an elliptic fibration.

We may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}$$

due to Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7.

The exceptional divisor E of the birational morphism α contains two singular points P and Q that are quotient singularity of types $\frac{1}{a_3-a_2}(1, a_2, a_3 - 2a_2)$ and $\frac{1}{a_2}(1, 1, a_3 - 2a_2)$, respectively. The base locus of $|-a_1K_X|$ consists of the irreducible curve C defined by $x = y = 0$. The base locus of $|-a_1K_U|$ consists of the proper transform C_U and the unique irreducible curve L in $|\mathcal{O}_{\mathbb{P}(1,a_2,a_3-a_2)}(1)|$ on the surface E.

Lemma 2.11.1. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q, then $\mathcal{M} = |-a_1K_X|$.*

Proof. Let $\pi : Y \to U$ be the Kawamata blow up at the point Q with weights $(1, 1, a_3 - 2a_2)$ and G_Q be its exceptional divisor. Then, the base locus of the pencil $|-a_1K_Y|$ consists of the irreducible curves C_Y and L_Y. Then, our situation is exactly same as Lemma 1.12.1. Using the same proof, we get $\mathcal{M} = |-a_1K_X|$. □

The exceptional divisor F of the birational morphism β contains two singular points P_1 and P_2 that are quotient singularities of types $\frac{1}{a_2}(1, 1, a_2 - 1)$ and $\frac{1}{a_3-2a_2}(1, 1, a_3 - 2a_2 - 1)$, respectively.

Lemma 2.11.2. *The set $\mathbb{CS}(W \frac{1}{n}\mathcal{M}_W)$ cannot contain the point P_2.*

Proof. Let $\sigma_2 : V_2 \to W$ be the Kawamata blow up at the point P_2. The proper transform $\mathcal{D}$ of the linear system $|-a_2K_X|$ consists of two irreducible curves $\bar{C}_{V_2}$ and L_{V_2}. Applying the same method as in Lemma 2.11.1 to the linear system $\mathcal{D}$, we obtain an absurd identity $\mathcal{M} = |-a_2K_X|$. □

Proposition 2.11.3. *The linear system $|-a_1K_X|$ is the only Halphen pencil on X.*

Proof. Lemma 0.2.7 implies that either $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) = \{P\}$ or $Q \in \mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$. The latter case implies $\mathcal{M} = |-a_1K_X|$ by Lemma 2.11.1. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point P. Then, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must contain the point P_1 by Lemma 2.11.2. Let $\sigma_1 : V_1 \to W$ be the Kawamata blow up at the point P_1. Then, the base locus of the pencil $|-a_1K_{V_1}|$ consists of the irreducible curves C_{V_1} and L_{V_1}. Applying the same method as in Lemma 2.11.1, we obtain $\mathcal{M} = |-a_1K_X|$. □

We suppose that $\daleth = 77$ or 85. The hypersurface $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ always contains the point $O = (0 : 0 : 0 : 1 : 0)$ as a quotient singularity of type $\frac{1}{a_3}(1, a_1, a_3 - a_1)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1, a_1, a_3 - a_1)$,
- β is the Kawamata blow up with weights $(1, a_1, a_3 - 2a_1)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{a_3 - a_1}(1, a_1, a_3 - 2a_1)$,
- η is an elliptic fibration.

As in the previous case, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}.$$

The exceptional divisor E of the birational morphism α contains two singular points P and Q that are quotient singularities of types $\frac{1}{a_3 - a_1}(1, a_1, a_3 - 2a_1)$ and $\frac{1}{a_1}(1, 1, a_1 - 1)$, respectively.

Unlike the previous case, we have the opposite statement for the point Q as follows:

Lemma 2.11.4. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain the point Q.*

Proof. Suppose the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q. Let $\pi : Y \to U$ be the Kawamata blow up at the point Q. The base locus of the proper transform $\mathcal{D}$ of the linear system $|-a_4K_X|$ does not contain any curve. Therefore, a general surface D in the linear system $\mathcal{D}$ is nef. However, we can easily check that $D \cdot M_1 \cdot M_2 < 0$ for general surfaces M_1 and M_2 in $\mathcal{M}_Y$. It is a contradiction. □

Proposition 2.11.5. *If $\daleth \in \{77, 85\}$, then the linear system $|-a_1K_X|$ is a unique Halphen pencil on X.*

Proof. The proof is the same as the cases $\daleth = 63$ and 83. The only difference is Lemma 2.11.1. We replace it by Lemma 2.11.4. □

2.12. Case $\daleth = 65$, hypersurface of degree 27 in $\mathbb{P}(1, 2, 5, 9, 11)$.

The threefold X is a general hypersurface of degree 27 in $\mathbb{P}(1, 2, 5, 9, 11)$ with $-K_X^3 = \frac{3}{110}$. The singularities of X consist of one singular point O that is a quotient singularity of type $\frac{1}{11}(1, 2, 9)$, one point of type $\frac{1}{5}(1, 4, 1)$, and four points of type $\frac{1}{2}(1, 1, 1)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,5), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,2,9)$,
- β is the Kawamata blow up with weights $(1,2,7)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{9}(1,2,7)$ contained in the exceptional divisor of the birational morphism α,
- γ is the Kawamata blow up with weights $(1,2,5)$ at the singular point of the variety W that is a quotient singularity of type $\frac{1}{7}(1,2,5)$ contained in the exceptional divisor of the birational morphism β,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{5}(1,4,1)$, then $\mathcal{M} = |-2K_X|$ by Lemma 0.3.11. Therefore, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$ by Lemma 0.3.3 and Corollary 0.3.8.

The hypersurface X can be given by the equation

$$w^2z + wf_{16}(x,y,z,t) + f_{27}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces cut on the hypersurface X by

$$\lambda x^5 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Even though the linear system $\mathcal{P}$ is not a Halphen pencil, it is helpful for our proof. Note that the base locus of the pencil $\mathcal{P}$ consists of the irreducible curve $\bar{C}$.

The exceptional divisor $E \cong \mathbb{P}(1,2,9)$ contains two singular points P and Q of U that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{9}(1,2,7)$, respectively. Let L be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,9)}(1)|$ on the surface E.

The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 0.2.4. Hence, either the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{D})$ contains the point P or it consists of the point Q by Lemma 0.2.7.

Lemma 2.12.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain the point P.*

Proof. Suppose that $P \in \mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$. Let $\pi_P : U_P \to U$ be the Kawamata blow up at the point P with weights $(1,1,1)$ and G_P be its exceptional divisor. Then, $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ holds by Lemma 0.2.6.

The base locus of the pencil $\mathcal{P}_{U_P}$ consists of the irreducible curves $\bar{C}_{U_P}$, L_{U_P}, and a line $\bar{L}$ on $G_P \cong \mathbb{P}^2$. For a general surface D_{U_P} of the pencil $\mathcal{P}_{U_P}$,

$$S_{U_P} \cdot D_{U_P} = \bar{C}_{U_P} + L_{U_P},\ E_{U_P} \cdot D_{U_P} = 5L_{U_P},\ G_P \cdot D_{U_P} = \bar{L}.$$

The surface D_{U_P} is normal. On the other hand, we have

$$\begin{cases} E_{U_P} \sim_{\mathbb{Q}} \pi_P^*(E) - \dfrac{1}{2}G_P, \\ D_{U_P} \sim_{\mathbb{Q}} (\alpha \circ \pi_P)^*(-5K_X) - \dfrac{5}{11}\pi_P^*(E) - \dfrac{1}{2}G_P, \\ S_{U_P} \sim_{\mathbb{Q}} (\alpha \circ \pi_P)^*(-K_X) - \dfrac{1}{11}\pi_P^*(E) - \dfrac{1}{2}G_P, \end{cases}$$

which implies

$$L_{U_P} \cdot L_{U_P} = -\frac{73}{450},\ \bar{C}_{U_P} \cdot L_{U_P} = \frac{4}{45},\ \bar{C}_{U_P} \cdot \bar{C}_{U_P} = -\frac{497}{550}$$

on the surface D_{U_P}. Therefore, the intersection form of the curves $\bar{C}_{U_P}$ and L_{U_P} on the surface D_{U_P} is negative-definite. On the other hand, we have

$$\mathcal{M}_{U_P}\Big|_{D_{U_P}} \equiv -nK_{U_P}\Big|_{D_{U_P}} \equiv nS_{U_P}\Big|_{D_{U_P}} \equiv n\bar{C}_{U_P} + nL_{U_P},$$

which implies $\mathcal{M}_{U_P} = \mathcal{P}_{U_P}$ by Theorem 0.2.9. Hence, $n = 5$, but $D_{U_P} \not\sim_{\mathbb{Q}} -5K_{U_P}$, which is a contradiction. □

Hence, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q. The exceptional divisor $F \cong \mathbb{P}(1,2,7)$ of β contains two singular points Q_1 and Q_2 of W that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{7}(1,2,5)$, respectively. Let $\tilde{L}$ be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)|$ on the surface F.

It follows from Theorem 0.2.4 that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty because the divisor $-K_W$ is nef and big.

Lemma 2.12.2. *The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ does not contain the point Q_1.*

Proof. Suppose that $Q_1 \in \mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$. Let $\pi_1 : W_1 \to W$ be the Kawamata blow up at the point Q_1 with weights $(1,1,1)$ and G_1 be its exceptional divisor.

The base locus of the pencil $\mathcal{P}_{W_1}$ consists of the irreducible curves $\bar{C}_{W_1}$, L_{W_1}, $\tilde{L}_{W_1}$, Δ_1, Δ_2, and Δ, where the curves Δ_1 and Δ_2 are the lines on $G_1 \cong \mathbb{P}^2$ cut out by the divisors E_{W_1} and F_{W_1}, respectively, and the curve Δ is a line on G_1 different from the lines Δ_1 and Δ_2.

For a general surface D_{W_1} in the pencil $\mathcal{P}_{W_1}$, we have

$$S_{W_1} \cdot D_{W_1} = C_{W_1} + L_{W_1} + \tilde{L}_{W_1},\ E_{W_1} \cdot D_{W_1} = 5L_{W_1} + \Delta_1,\ F_{W_1} \cdot D_{W_1} = 5\tilde{L}_{W_1} + \Delta_2.$$

The surface D_{W_1} is normal and it is smooth in a neighborhood of G_1. In particular, it follows from the local computations and the Adjunction formula that the equalities

$$\Delta_1 \cdot \Delta_2 = \Delta_1 \cdot \tilde{L}_{W_1} = \Delta_2 \cdot L_{W_1} = 1,\ \Delta_1 \cdot C_{W_1} = \Delta_2 \cdot C_{W_1} = 0,\ \Delta_1^2 = \Delta_2^2 = -4$$

hold on the surface D_{W_1}. However, we have

$$\begin{cases} F_{W_1} \sim_{\mathbb{Q}} \pi_1^*(F) - \dfrac{1}{2}G, \\ E_{W_1} \sim_{\mathbb{Q}} (\beta \circ \pi_1)^*(E) - \dfrac{7}{9}\pi_1^*(F) - \dfrac{1}{2}G_1, \\ D_{W_1} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi_1)^*(-5K_X) - \dfrac{5}{11}(\beta \circ \pi_1)^*(E) - \dfrac{5}{9}\pi_1^*(F) - \dfrac{3}{2}G_1, \\ S_{W_1} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi_1)^*(-K_X) - \dfrac{1}{11}(\beta \circ \pi_1)^*(E) - \dfrac{1}{9}\pi_1^*(F) - \dfrac{1}{2}G_1. \end{cases}$$

We can then obtain

$$C_{W_1} \cdot C_{W_1} = L_{W_1} \cdot L_{W_1} = -\frac{1}{2},\ \tilde{L}_{W_1} \cdot \tilde{L}_{W_1} = -\frac{3}{7},\ C_{W_1} \cdot L_{W_1} = C_{W_1} \cdot \tilde{L}_{W_1} = L_{W_1} \cdot \tilde{L}_{W_1} = 0$$

on the surface D_{W_1}. Therefore, the intersection form of the curves C_{W_1}, L_{W_1}, and $\tilde{L}_{W_1}$ on the surface D_{W_1} is negative-definite. On the other hand, we have

$$\mathcal{M}_{W_1}\Big|_{D_{W_1}} \equiv -nK_{W_1}\Big|_{D_{W_1}} \equiv nS_{W_1}\Big|_{D_{W_1}} \equiv nC_{W_1} + nL_{W_1} + n\tilde{L}_{W_1},$$

which implies $\mathcal{M}_{W_1} = \mathcal{P}_{W_1}$ by Theorem 0.2.9. Hence, we have $n = 5$, but $D_{W_1} \not\equiv -5K_{W_1}$, which is a contradiction. □

Thus, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ consists of the point Q_2 by Lemma 0.2.7.

The exceptional divisor of the birational morphism γ contains two singular points O_1 and O_2 of Y that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{5}(1,2,3)$, respectively. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must contain either the point O_1 or the point O_2 by Theorem 0.2.4 and Lemma 0.2.7.

Lemma 2.12.3. *The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain the point O_1.*

Proof. Suppose that $O_1 \in \mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$. Let $\sigma_1 : V_1 \to Y$ be the Kawamata blow up at the point O_1 with weights $(1,1,1)$ and H_1 be the exceptional divisor of the birational morphism σ_1.

A general surface D_{V_1} in $\mathcal{P}_{V_1}$ is normal and

$$\mathcal{M}_{V_1}\Big|_{D_{V_1}} \equiv -nK_{V_1}\Big|_{D_{V_1}} \equiv nS_{V_1}\Big|_{D_{V_1}}.$$

The intersection $D_{V_1} \cap H_1$ is a line on $G_1 \cong \mathbb{P}^2$ that is different from the line $S_{V_1} \cap G_1$. However, the curve $D_{V_1} \cap H_1$ is a fiber of the elliptic fibration $\eta \circ \sigma_1|_{D_{V_1}}$ over the point $\eta \circ \sigma_1(S_{V_1} \cap H_1)$. The support of the cycle $S_{V_1} \cdot D_{V_1}$ contains all components of the fiber of the elliptic fibration $\eta \circ \sigma_1|_{D_{V_1}}$ over the point $\eta \circ \sigma_1(S_{V_1} \cap H_1)$ that are different from the curve $D_{V_1} \cap H_1$. Hence, the intersection form of the components of the cycle $S_{V_1} \cdot D_{V_1}$ are negative-definite on the surface D_{V_1}, which implies that $\mathcal{M}_{V_1} = \mathcal{P}_{V_1}$ by Theorem 0.2.9. Hence, we have $n = 5$, but it follows from explicit calculations that

$$D_{V_1} \sim_{\mathbb{Q}} \sigma_1^*(-5K_Y) - \frac{1}{2}H_1 \not\sim_{\mathbb{Q}} -5K_{V_1},$$

which is a contradiction. □

Proposition 2.12.4. *The linear system $|-2K_X|$ is a unique Halphen pencil on X.*

Proof. By what we have proved so far, we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point O_2. Let $\sigma_2 : V_2 \to Y$ be the Kawamata blow up at the point O_1 with weights $(1,2,3)$. Then, $|-2K_{V_2}|$ is the proper transform of the pencil $|-2K_X|$. Its base locus consists of the irreducible curve C_{V_2}. Because $\mathcal{M}_{V_1} \sim_{\mathbb{Q}} -nK_{V_1}$ and $-K_{V_1} \cdot C_{V_2} < 0$, Theorem 0.2.9 implies that $\mathcal{M} = |-2K_X|$. □

2.13. Cases $\gimel = 72$, 89, 90, 92, and 94.

Suppose that $\gimel \in \{72, 89, 90, 92, 94\}$. Then, the threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ always contains a quotient singularity O of type $\frac{1}{a_1+a_2}(1, a_1, a_2)$.

We also have a commutative diagram as follows:

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2) \end{tikzcd}$$

where

- ψ is a natural projection,
- π is the Kawamata blow up at the point O with weights $(1, a_1, a_2)$,
- η is an elliptic fibration.

We may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the singular point O due to Lemmas 0.3.3, 0.3.10, 0.3.11 and Corollary 0.3.7.

The exceptional divisor E of the birational morphism π contains two singular points P and Q of types $\frac{1}{a_1}(1, a, a_1 - a)$ and $\frac{1}{a_2}(1, a_1, a_2 - a_1)$, respectively, where $a = |2a_1 - a_2|$. Then, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains either the singular point P or the singular point Q by Lemma 0.2.7.

Lemma 2.13.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain the point P.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P. Let $\alpha : U_P \to U$ be the Kawamata blow up at the point P with weights $(1, a, a_1 - a)$ and G be the exceptional divisor of the birational morphism α. Then,

$$\mathcal{M}_{U_P} \sim_{\mathbb{Q}} (\pi \circ \alpha)^*\Big(-nK_X\Big) - \frac{n}{a_1 + a_2}\alpha^*(E) - \frac{n}{a_1}G \sim_{\mathbb{Q}} -nK_{U_P}$$

by Lemma 0.2.6.

Let $\mathcal{D}$ be the proper transform of the linear system $|-a_2K_X|$ by the birational morphism $\pi\circ\alpha$. Then, $S_{U_P}\sim_{\mathbb{Q}}-K_{U_P}$ and

$$\mathcal{D}\sim_{\mathbb{Q}}\big(\pi\circ\alpha\big)^*\Big(-a_2K_X\Big)-\frac{a_2}{a_1+a_2}\alpha^*\big(E\big)-\frac{1}{a_1}G,$$

but the base locus of the linear system $\mathcal{D}$ consists of the irreducible curve $\bar{C}_{U_P}$ whose image to X is the base curve of the linear system $|-a_2K_X|$. Let D be a general surface in $\mathcal{D}$ and M be a general surface in $\mathcal{M}_{U_P}$. The surface D is normal. We have $\bar{C}_{U_P}^2<0$ on the normal surface D and $M|_D\equiv n\bar{C}_{U_P}$, which implies that $\mathcal{M}_{U_P}=\mathcal{D}$ by Theorem 0.2.9. However, the linear system $\mathcal{D}$ is not a pencil. □

Proposition 2.13.2. *If* $\gimel=72,89,90,92,94$*, then* $\mathcal{M}=|-a_1K_X|$.

Proof. By Lemma 2.13.1, we may assume that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains the point Q.

Let $\beta:W\to U$ be the Kawamata blow up of the point Q with weights $(1,a_1,a_2-a_1)$. The linear system $|-a_1K_W|$ is the proper transform of the pencil $|-a_1K_X|$ and the base locus of the pencil $|-a_1K_W|$ consists of the irreducible curve C_W whose image to X is the base curve of $|-a_1K_X|$. Then, the inequality $-K_W\cdot C_W<0$ and the equivalence $\mathcal{M}_W\sim_{\mathbb{Q}}-nK_W$ imply that the pencil $\mathcal{M}_W$ coincides with the pencil $|-a_1K_W|$ by Theorem 0.2.9. □

2.14. Cases $\gimel=75$ and 87.

In the case of $\gimel=75$, the threefold X is a general hypersurface of degree 30 in $\mathbb{P}(1,4,5,6,15)$ with $-K_X^3=\frac{1}{60}$. Its singularities consist of one quotient singular point of type $\frac{1}{4}(1,1,3)$, one quotient singular point of type $\frac{1}{3}(1,1,2)$, two quotient singular points of type $\frac{1}{2}(1,1,1)$, and two quotient singular points of type $\frac{1}{5}(1,4,1)$.

In the case of $\gimel=87$, the threefold X is a general hypersurface of degree 40 in $\mathbb{P}(1,5,7,8,20)$ with $-K_X^3=\frac{1}{140}$. It has one quotient singular point of type $\frac{1}{4}(1,1,3)$, two quotient singular points of type $\frac{1}{5}(1,2,3)$, and one quotient singular point of type $\frac{1}{7}(1,1,6)$.

In both cases, the threefold X cannot be birationally transformed to an elliptic fibration ([4]). However, it can be rationally fibred by K3 surfaces.

Proposition 2.14.1. *If* $\gimel\in\{75,87\}$*, then* $|-a_1K_X|$ *is a unique Halphen pencil on* X.

Proof. Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7 immediately imply the result. □

Part 3. Fano threefold hypersurfaces with two Halphen pencils.

3.1. Case $\daleth = 45$, hypersurface of degree 20 in $\mathbb{P}(1,3,4,5,8)$.

Let X be the hypersurface given by a general quasihomogeneous equation of degree 18 in $\mathbb{P}(1,3,4,5,8)$ with $-K_X^3 = \frac{1}{24}$. Then, the singularities of X consist of one singular point P that is a quotient singularity of type $\frac{1}{8}(1,3,5)$, one singular point of type $\frac{1}{3}(1,1,2)$, and two points of type $\frac{1}{4}(1,3,1)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,3,4), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is a blow up at the singular point P with weights $(1,3,5)$,
- β is the blow up with weights $(1,3,2)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,3,2)$,
- η is an elliptic fibration.

Also, by the generality of the hypersurface, we may assume that the hypersurface X is defined by the equation

$$w^2z + wf_{12}(x,y,z,t) + f_{20}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Proposition 0.3.12 implies that the linear system $|-3K_X|$ is a Halphen pencil. Let $\mathcal{P}$ be the pencil on X given by the equations

$$\lambda x^4 + \mu z = 0,$$

where $(\lambda,\mu) \in \mathbb{P}^1$. We see that the linear system $\mathcal{P}$ is another Halphen pencil on X.

K3-Proposition 3.1.1. *A general member of the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. It is a compactification of a double cover of $\mathbb{C}^2$ ramified along a sextic curve. It cannot be a rational surface by Theorem 0.1.3. Therefore, it is birational to a smooth K3 surface. $\square$

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains one of the singular points of type $\frac{1}{4}(1,3,1)$, then $\mathcal{M} = |-3K_X|$ by Lemma 0.3.11. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{P\Big\}$$

by Lemma 0.3.3 and Corollary 0.3.7.

The exceptional divisor $E \cong \mathbb{P}(1,3,5)$ of the birational morphism α contains two singular points P_1 and P_2 of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{5}(1,3,2)$, respectively. For the convenience, let L be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,5)}(1)|$ on the surface E.

Lemma 3.1.2. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ cannot contain the point P_1.*

Proof. Suppose so. Then, we consider the Kawamata blow up $\alpha_1 : V \to U$ at the point P_1 with weights $(1,1,2)$. Let $\mathcal{D}_V$ be the proper transform, by the birational morphism $\alpha \circ \alpha_1$, of the linear system $\mathcal{D}$ on X defined by the equations $\lambda_0 x^5 + \lambda_1 x^2 y + \lambda_2 t = 0$, where $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$. Then, its base locus consists of the irreducible curve $\tilde{C}_V$ whose image to X is the base curve of the linear system $\mathcal{D}$. We easily see that $S_V \cdot D_V = \tilde{C}_V$ and

$$D_V \sim_{\mathbb{Q}} (\alpha \circ \alpha_1)^*\Big(-5K_X\Big) - \frac{5}{8}\alpha_1^*(E) - \frac{1}{3}F_1,$$

$$S_V \sim_{\mathbb{Q}} (\alpha \circ \alpha_1)^*\Big(-K_X\Big) - \frac{1}{8}\alpha_1^*(E) - \frac{1}{3}F_1,$$

where D_V is a general surface in $\mathcal{D}_V$ and F_1 is the exceptional divisor of α_1. Since the curve $\tilde{C}_V$ has negative self-intersection on the normal surface D_V, Theorem 0.2.9 implies $\mathcal{M} = \mathcal{D}$. But this is absurd because $\mathcal{D}$ is not a pencil. □

Therefore, we may assume that $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_W) = \{P_2\}$. The exceptional divisor F of the birational morphism β contains two singular points Q_1 and Q_2 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$, respectively.

Lemma 3.1.3. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. Let $\beta_1 : W_1 \to W$ be the Kawamata blow up at the point Q_1 with weights $(1,1,2)$. The base locus of $\mathcal{P}_{W_1}$ consists of the irreducible curve C_{W_1} and the irreducible curve L_{W_1}. A general surface D_{W_1} in $\mathcal{P}_{W_1}$ is normal and the intersection form of the curves C_{W_1} and L_{W_1} is negative-definite on the surface D_{W_1}. Because $\mathcal{M}_{W_1}|_{D_{W_1}} \equiv nC_{W_1} + nL_{W_1}$, we obtain $\mathcal{M} = \mathcal{P}$ from Theorem 0.2.9. □

Lemma 3.1.4. *The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ cannot contain the point Q_2.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2. Let $\beta_2 : W_2 \to W$ be the Kawamata blow up at the point Q_2. We then consider the proper transform $\mathcal{L}_{W_2}$ of the pencil $|-3K_X|$ by the birational morphism $\alpha \circ \beta \circ \beta_2$. For a general surface D_{W_2} in $\mathcal{L}_{W_2}$, we have

$$D_{W_2} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \beta_2)^*\Big(-3K_X\Big) - \frac{3}{8}(\beta \circ \beta_2)^*(E) - \frac{3}{5}\beta_2^*(F) - \frac{1}{2}G,$$

where G is the exceptional divisor of β_2. Also, we have

$$S_{W_2} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \beta_2)^*\Big(-K_X\Big) - \frac{1}{8}(\beta \circ \beta_2)^*(E) - \frac{1}{5}\beta_2^*(F) - \frac{1}{2}G.$$

Since $\mathcal{M}_{W_2}|_{D_{W_2}} \equiv nC_{W_2}$, the surface D_{W_2} is normal, and the self-intersection number $C_{W_2}^2$ on the normal surface D_{W_2} is $-\frac{1}{2}$, we obtain the identity $\mathcal{M}_{W_2} = \mathcal{L}_{W_2}$ from Theorem 0.2.9. However, $D_{W_2} \not\sim_{\mathbb{Q}} -3K_{W_2}$. It is a contradiction. □

Proposition 3.1.5. *The linear systems $|-3K_X|$ or $\mathcal{P}$ are the only Halphen pencils on X.*

Proof. We apply Lemma 0.3.11 to the singular points on X of type $\frac{1}{4}(1,3,1)$ and the lemmas above to the singular point P. □

3.2. Case $\gimel = 48$, hypersurface of degree 21 in $\mathbb{P}(1,2,3,7,9)$.

In the case of $\gimel = 48$, the hypersurface X is defined by a general quasihomogeneous equation of degree 21 in $\mathbb{P}(1,2,3,7,9)$ with $-K_X^3 = \frac{1}{18}$. It has a quotient singularity of type $\frac{1}{9}(1,2,7)$ at the point $O = (0:0:0:0:1)$. It also has one quotient singular point of type $\frac{1}{2}(1,1,1)$ and two quotient singular points of type $\frac{1}{3}(1,2,1)$.

We have an elliptic fibration as follows:

$$\begin{tikzcd} Y \arrow[d, "\alpha"'] & U \arrow[l, "\beta"'] & V \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,2,7)$,
- β is the Kawamata blow up with weights $(1,2,5)$ at the singular point of Y that is a quotient singularity of type $\frac{1}{7}(1,2,5)$,
- γ is the Kawamata blow up with weights $(1,2,3)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{5}(1,2,3)$,

- η is an elliptic fibration.

Proposition 0.3.12 implies that the linear system $|-2K_X|$ is a Halphen pencil. However, we have another Halphen pencil. The hypersurface X can be given by the equation

$$w^2z + wf_{12}(x,y,z,t) + f_{21}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil on X given by the equations

$$\lambda x^3 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. We will see that the linear system $\mathcal{P}$ is a Halphen pencil on X (K3-Proposition 3.2.3).

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a singular point of type $\frac{1}{3}(1,2,1)$ on X, then the identity $\mathcal{M} = |-2K_X|$ follows from Lemma 0.3.11. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}$$

due to Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, and Corollary 0.3.7.

The exceptional divisor $E \cong \mathbb{P}(1,2,7)$ of the birational morphism α contains two quotient singular points P_1 and Q_1 of types $\frac{1}{7}(1,2,5)$ and $\frac{1}{2}(1,1,1)$, respectively. For the convenience, let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)|$ on the surface E.

Lemma 3.2.1. *The set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ cannot contain the point Q_1.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q_1. Let $\alpha_1 : W_1 \to Y$ be the Kawamata blow up at the point Q_1 with weights $(1,1,1)$. By Lemma 0.2.6, we have $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$.

We consider the linear system $|-7K_X|$ on X that has no base curves. Also, the proper transform $\mathcal{D}_{W_1}$ of the linear system $|-7K_X|$ by the birational $\alpha \circ \alpha_1$ has no base curve. Let D_1 be a general member in $\mathcal{D}_{W_1}$. The divisor D_1 is nef. Meanwhile, we have

$$D_1 \sim_{\mathbb{Q}} (\alpha \circ \alpha_1)^*(-7K_X) - \frac{7}{9}\alpha_1^*(E) - \frac{1}{2}E_1,$$

where E_1 is the exceptional divisor of α_1. Therefore, $D_1 \cdot M_1 \cdot M_2 = -\frac{n^2}{6} < 0$, where M_1 and M_2 are general members in $\mathcal{M}_{W_1}$. It is a contradiction. $\square$

Therefore, we may assume that $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) = \{P_1\}$. The exceptional divisor $F \cong \mathbb{P}(1,2,5)$ of the birational morphism β contains two singular points P_2 and Q_2 that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$, respectively. The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains either the point P_2 or the point Q_2. For the convenience, we denote the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,5)}(1)|$ on the surface F by $\bar{L}$.

Lemma 3.2.2. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q_2, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q_2. Let $\beta_2 : W_2 \to U$ be the Kawamata blow up at the point Q_2 with weights $(1,1,1)$. Let F_2 be the exceptional divisor of β_2. Then,

$$\mathcal{M}_{W_2} \sim_{\mathbb{Q}} -nK_{W_2} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \beta_2)^*(-nK_X) - \frac{n}{9}(\beta \circ \beta_2)^*(E) - \frac{n}{7}\beta_2^*(F) - \frac{n}{2}F_2.$$

The base locus of the pencil $\mathcal{P}_{W_2}$ consists of three irreducible curves $\bar{C}_{W_2}$, L_{W_2}, and $\bar{L}_{W_2}$.

Let D_{W_2} be a general surface in the pencil $\mathcal{P}_{W_2}$. Then, the surface D_2 is normal and we have

$$\begin{cases} D_{W_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\beta_2)^*(-3K_X) - \dfrac{3}{9}(\beta\circ\beta_2)^*(E) - \dfrac{3}{7}\beta_2^*(F) - \dfrac{3}{2}F_2, \\ S_{W_2} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\beta_2)^*(-K_X) - \dfrac{1}{9}(\beta\circ\beta_2)^*(E) - \dfrac{1}{7}\beta_2^*(F) - \dfrac{1}{2}F_2, \\ E_{W_2} \sim_{\mathbb{Q}} (\beta\circ\beta_2)^*(E) - \dfrac{5}{7}\beta_2^*(F) - \dfrac{1}{2}F_2, \\ F_{W_2} \sim_{\mathbb{Q}} \beta_2^*(F) - \dfrac{1}{2}F_2. \end{cases}$$

Furthermore,

$$S_{W_2}\cdot D_{W_2} = \bar{C}_{W_2} + L_{W_2} + \bar{L}_{W_2}, \quad E_{W_2}\cdot D_{W_2} = 3L_{W_2}, \quad F_{W_2}\cdot D_{W_2} = 3\bar{L}_{W_2}.$$

Consider the curves $\bar{C}_{W_2}$, L_{W_2}, and $\bar{L}_{W_2}$ as divisors on D_{W_2}. Then, the equivalences above imply that

$$\bar{C}_{W_2}\cdot\bar{C}_{W_2} = L_{W_2}\cdot L_{W_2} = -\frac{1}{2}, \quad \bar{L}_{W_2}\cdot\bar{L}_{W_2} = -\frac{2}{5}, \quad C_{W_2}\cdot L_{W_2} = C_{W_2}\cdot\bar{L}_{W_2} = L_{W_2}\cdot\bar{L}_{W_2} = 0.$$

Therefore, the intersection form of these curves on D_{W_2} is negative-definite. On the other hand, we have

$$M_{W_2}\Big|_{D_{W_2}} \equiv -nK_{W_2}\Big|_{D_{W_2}} \equiv nS_{W_2}\Big|_{D_{W_2}} \equiv n\bar{C}_{W_2} + nL_{W_2} + n\bar{L}_{W_2},$$

where M_{W_2} is a general surface in the pencil $\mathcal{M}_{W_2}$. Therefore, we obtain $\mathcal{M} = \mathcal{P}$ from Theorem 0.2.9. □

K3-Proposition 3.2.3. *A general surface in the pencil $\mathcal{P}$ is birational to a K3 surface.*

Proof. We use the same notations in the proof of Lemma 3.2.2. Note that the proof of Lemma 3.2.2 and Proposition 0.3.12 shows the linear system $\mathcal{P}$ is a Halphen pencil.

Suppose that the intersection curve $\Delta = F_2\cdot D_{W_2}$ is a smooth curve on $F_2 \cong \mathbb{P}^2$. Because the degree of the curve Δ on F_2 is three, it must be an elliptic curve. One can see that the singularities of the image surface $D_U := \beta_2(D_{W_2})$ are rational except the point Q_2. Moreover, the restricted morphism $\beta_2|_{D_{W_2}} : D_{W_2} \to D_U$ resolves the singular point Q_1. Therefore, the singularities of the surface D_{W_2} is rational. Because the divisor $-K_U$ is nef and big, the Leray spectral sequence for the morphism $\bar{\beta} := \beta_2|_{D_{W_2}} : D_{W_2} \to D_U$ shows

$$0 \to H^1(D_U, \mathcal{O}_{D_U}) = 0 \to H^1(D_{W_2}, \mathcal{O}_{D_{W_2}}) \to H^0(D_U, R^1\bar{\beta}_*\mathcal{O}_{D_{W_2}}) \to H^2(D_U, \mathcal{O}_{D_U}) = 0,$$

and hence the irregularity $h^1(D_{W_2}, \mathcal{O}_{D_{W_2}}) = 1$. Because the surface D_{W_2} has only rational singularities, a smooth surface birational to the surface D_{W_2} has the same irregularity. However, the surface D_{W_2} is birational to a K3 surface or an abelian surface by Corollary 0.2.11, which is a contradiction. Therefore, the curve Δ must be a singular curve, and hence a rational curve. Therefore, Corollary 0.2.12 completes the proof. □

Due to the lemma above, we may assume that $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) = \{P_2\}$. Let $\gamma : V \to U$ be the Kawamata blow up at the point P_2 with weights $(1,2,3)$. Then, the exceptional divisor $G \cong \mathbb{P}(1,2,3)$ of the birational morphism γ contains two singular points P_3 and Q_3 that are quotient singularities of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{2}(1,1,1)$, respectively. Again, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ must contain either the point P_3 or the point Q_3.

Lemma 3.2.4. *The set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ cannot contain the point Q_3.*

Proof. Suppose that the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ contains the point Q_3. Let $\gamma_3 : W_3 \to V$ be the Kawamata blow up at the point Q_3 with weights $(1,1,1)$ and G_3 be the exceptional divisor of γ_3. Then, $\mathcal{M}_{W_3} \sim_{\mathbb{Q}} -nK_{W_3}$ by Lemma 0.2.6. The base locus of the pencil $\mathcal{P}_{W_3}$ consists of three irreducible curves $\bar{C}_{W_3}$, L_{W_3}, and $\bar{L}_{W_3}$.

Let D_{W_3} be a general surface in the pencil $\mathcal{P}_{W_3}$. Then, the surface D_{W_3} is normal and we have

$$\begin{cases} D_{W_3} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma\circ\gamma_3)^*(-3K_X) - \dfrac{3}{9}(\beta\circ\gamma\circ\gamma_3)^*(E) - \dfrac{3}{7}(\gamma\circ\gamma_3)^*(F) - \dfrac{3}{5}\gamma_3^*(G) - \dfrac{1}{2}G_3 \\ S_{W_3} \sim_{\mathbb{Q}} (\alpha\circ\beta\circ\gamma\circ\gamma_3)^*(-K_X) - \dfrac{1}{9}(\beta\circ\gamma\circ\gamma_3)^*(E) - \dfrac{1}{7}(\gamma\circ\gamma_3)^*(F) - \dfrac{1}{5}\gamma_3^*(G) - \dfrac{1}{2}G_3, \\ E_{W_3} \sim_{\mathbb{Q}} (\beta\circ\gamma\circ\gamma_3)^*(E) - \dfrac{5}{7}(\gamma\circ\gamma_3)^*(F) - \dfrac{3}{5}\gamma_3^*(G) - \dfrac{1}{2}G_3, \\ F_{W_3} \sim_{\mathbb{Q}} (\gamma\circ\gamma_3)^*(F) - \dfrac{3}{5}\gamma_3^*(G) - \dfrac{1}{2}G_3, \\ G_{W_3} \sim_{\mathbb{Q}} \gamma_3^*(G) - \dfrac{1}{2}G_3, \end{cases}$$

Furthermore,

$$S_{W_3}\cdot D_{W_3} = \bar{C}_{W_3} + L_{W_3} + \bar{L}_{W_3}, \quad E_{W_3}\cdot D_{W_3} = 3L_{W_3}, \quad F_{W_3}\cdot D_{W_3} = 3\bar{L}_{W_3}.$$

Consider the curves $\bar{C}_{W_3}$, L_{W_3}, and $\bar{L}_{W_3}$ as divisors on D_{W_3}. Then, the equivalences above imply that

$$\begin{gathered} \bar{C}_{W_3}\cdot\bar{C}_{W_3} = L_{W_3}\cdot L_{W_3} = -\frac{44}{90}, \quad \bar{L}_{W_3}\cdot\bar{L}_{W_3} = -\frac{35}{90}, \\ C_{W_3}\cdot L_{W_3} = C_{W_3}\cdot\bar{L}_{W_3} = \frac{19}{90}, \quad L_{W_3}\cdot\bar{L}_{W_3} = \frac{1}{90}. \end{gathered}$$

It is easy to see that the intersection form of these curves on D_{W_3} is negative-definite. On the other hand, we have

$$M_{W_3}\Big|_{D_{W_3}} \equiv -nK_{W_3}\Big|_{D_3} \equiv nS_{W_3}\Big|_{D_{W_3}} \equiv n\bar{C}_{W_3} + nL_{W_3} + n\bar{L}_{W_3},$$

where M_{W_3} is a general surface in the pencil $\mathcal{M}_{W_3}$. Therefore, we obtain $\mathcal{M}_{W_3} = \mathcal{P}_{W_3}$ from Theorem 0.2.9. However, it is a contradiction because $D_3 \not\sim_{\mathbb{Q}} -3K_{W_3}$. □

Therefore, the set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ consists of only one point P_3. Let $\delta : V_3 \to V$ be the Kawamata blow up at the point P_3 with weights $(1,2,1)$. The pencil $|-2K_{V_3}|$ is the proper transform of the pencil $|-2K_X|$. It has only one base curve C_{V_3} whose image to X is the base curve of the pencil $|-2K_X|$. Then, the inequality $-K_{V_3}\cdot C_{V_3} = -2K_{V_3}^3 < 0$ and the equivalence $\mathcal{M}_{V_3} \sim_{\mathbb{Q}} -nK_{V_3}$ imply $\mathcal{M} = |-2K_X|$ by Theorem 0.2.9.

Proposition 3.2.5. *If $\gimel = 48$, then the linear systems $|-2K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

3.3. Cases $\gimel = 55$, 80, and 91.

The threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum a_i$ always contains the point $O = (0:0:0:1:0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, a_1, a_3 - a_1)$. The threefold X can be given by

$$t^3 z + \sum_{i=0}^{2} t^i f_{d-ia_3}\big(x, y, z, w\big) = 0,$$

where f_i is a general quasihomogeneous polynomial of degree i.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2), \end{tikzcd}$$

where

- α is the Kawamata blow up at the point O with weights $(1, a_1, a_3 - a_1)$,
- β is the Kawamata blow up with weights $(1, a_1, a_3 - 2a_1)$ at the singular point Q of the variety U that is a quotient singularity of type $\frac{1}{a_3 - a_1}(1, a_1, a_3 - 2a_1)$,
- η is an elliptic fibration.

Let $\mathcal{P}$ be the pencil defined by

$$\lambda x^{a_2} + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Note that the base curve $\bar{C}$ of $\mathcal{P}$ is given by the equations $x = z = 0$. We will see that the linear systems $|-a_1 K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.

We may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}.$$

due to Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7.

The exceptional divisor $E \cong \mathbb{P}(1, a_1, a_3 - a_1)$ of the birational morphism α contains two singular points P and Q that are quotient singularities of types $\frac{1}{a_1}(1, a_1 - 1, 1)$ and $\frac{1}{a_3 - a_1}(1, a_1, a_3 - 2a_1)$, respectively. For the convenience, let L be the unique irreducible curve contained in $|\mathcal{O}_{\mathbb{P}(1,\, a_1,\, a_3 - a_1)}(1)|$ on the surface E.

Lemma 3.3.1. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P, then $\mathcal{M} = \mathcal{P}$.*

Proof. We will consider only the case $\daleth = 91$. The other cases can be shown by the same method.

Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P. Let $\gamma : V \to U$ be the Kawamata blow up at the point P with weights $(1, 3, 1)$ and let G_P be the exceptional divisor of γ.

Around the point O, the monomials x, y, and w can be considered as weighted local coordinates with weights $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 4$, and $\mathrm{wt}(w) = 9$. Therefore, around the singular point P, the birational morphism α is given by the equations

$$x = \tilde{x}\tilde{y}^{\frac{1}{13}},\ y = \tilde{y}^{\frac{4}{13}},\ w = \tilde{w}\tilde{y}^{\frac{9}{13}},$$

where $\tilde{x}$, $\tilde{y}$, and $\tilde{w}$ are weighted local coordinates with $\mathrm{wt}(\tilde{x}) = \mathrm{wt}(\tilde{w}) = 1$ and $\mathrm{wt}(\tilde{y}) = 3$. Let R be a general surface of the pencil $\mathcal{P}$. Then, R is given by the equation of the form

$$\lambda x^5 + \mu\Big(\delta_1 w^2 + \sum_{i=0}^{4} \delta_{2+i} x^{18-4i} y^i + \delta_7 x y^2 w + \delta_8 x^5 y w + \delta_9 x^9 w + \text{higher terms}\Big) = 0$$

near the point O, where $\delta_i \in \mathbb{C}$. The proper transform R_U is given by

$$\lambda \tilde{x}^5 + \mu\tilde{y}\Big(\delta_1 \tilde{w}^2 + \delta_2 \tilde{x}^2 + \delta_7 \tilde{x}\tilde{w} + \text{higher terms}\Big) = 0$$

near the point P. The base locus of the pencil $\mathcal{P}_U$ consists of the irreducible curves $\bar{C}_U$ and L. It shows that $|-5K_V| = \mathcal{P}_V$ and the base locus of the pencil $|-5K_V|$ consists of the curves $\bar{C}_V$ and L_V. Furthermore,

$$R_V \cdot S_V = \bar{C}_V + L_V,\ R_V \cdot E_V = 5L_V,$$

which implies that R_V is normal. On the other hand, we have

$$\begin{cases} R_V \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\Big(-5K_X\Big) - \dfrac{5}{13}\gamma^*(E) - \dfrac{5}{4}G_P, \\ S_V \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*\Big(-K_X\Big) - \dfrac{1}{13}\gamma^*(E) - \dfrac{1}{4}G_P, \\ E_V \sim_{\mathbb{Q}} \gamma^*(E) - \dfrac{3}{4}G_P, \end{cases}$$

which implies that on the normal surface R_V we have

$$\begin{cases} L_V \cdot L_V = \dfrac{E_V \cdot E_V \cdot R_V}{25} = -\dfrac{4}{45}, \\ \bar{C}_V \cdot \bar{C}_V = S_V \cdot S_V \cdot R_V - \dfrac{2}{5} S_V \cdot E_V \cdot R_V - \dfrac{E_V \cdot E_V \cdot R_V}{25} = -\dfrac{11}{30}, \\ \bar{C}_V \cdot L_V = \dfrac{S_V \cdot E_V \cdot R_V}{5} - \dfrac{E_V \cdot E_V \cdot R_V}{25} = \dfrac{1}{30}. \end{cases}$$

It immediately implies that the intersection form of the curves $\bar{C}_V$ and L_V on the normal surface R_V is negative-definite. On the other hand, we have

$$\mathcal{M}_V\Big|_{R_V} \equiv -nK_V\Big|_{R_V} \equiv nS_V\Big|_{R_V} \equiv n\bar{C}_V + nL_V,$$

which implies that $\mathcal{M}_V = |-5K_V|$ by Theorem 0.2.9. In particular, we have $\mathcal{M} = \mathcal{P}$. □

K3-Proposition 3.3.2. *A general surface in $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. We use the same notations in the proof of Lemma 3.3.1. The exceptional divisor G_P of the Kawamata blow up γ is isomorphic to $\mathbb{P}(1, a_1 - 1, 1)$. Then, the intersection $\Delta := G_P \cdot R_V$ is a curve of degree $a_2 = a_1 + 1$ on $\mathbb{P}(1, a_1 - 1, 1)$.

In the case $\daleth = 91$, the surface G_P has a quotient singular point. One can easily check the curve $G_P \cdot R_V$ passes through the singular point, and hence it is a rational curve. Therefore, Corollary 0.2.12 implies that the surface R_V is birational to a smooth K3 surface.

In the cases $\daleth = 55$ and 80, the curve Δ does not pass through a singular point of the surface G_P. We suppose that the curve Δ is smooth. Because it does not pass through any singular point of G_P and its degree on G_P is $a_2 = a_1 + 1$, it is an elliptic curve. One can see that the singularities of the image surface $R_U := \gamma(R_V)$ are rational except the point P. Then, the same argument of K3-Proposition 3.2.3 leads us to a contradiction. Therefore, the surface R_V is birational to a smooth K3 surface. □

The exceptional divisor F of the birational morphism β contains two singular points Q_1 and Q_2 that are quotient singularities of types $\frac{1}{a_1}(1, a_1 - 1, 1)$ and $\frac{1}{a_3 - 2a_1}(1, a_1, a_3 - 3a_1)$, respectively.

Lemma 3.3.3. *The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ does not contain the point Q_1.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. Let $\gamma_1 : Y_1 \to W$ be the Kawamata blow up at the singular point Q_1 with weights $(1, a_1 - 1, 1)$ and let G_1 be the exceptional divisor of γ_1.

Simple calculations imply that the base locus of the proper transform of the linear system $|-a_2K_X|$ on the threefold Y_1 consists of the curves $\bar{C}_{Y_1}$ and L_{Y_1}.

Let B be a general surface of the linear system $|-a_2K_X|$. Then,

$$B_{Y_1} \cdot S_{Y_1} = \bar{C}_{Y_1} + L_{Y_1},\ B_{Y_1} \cdot E_{Y_1} = L_{Y_1},$$

which implies that B_{Y_1} is normal. On the other hand, we have

$$\begin{cases} B_{Y_1} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \gamma_1)^*\Big(-a_2K_X\Big) - \dfrac{a_2}{a_3}(\beta \circ \gamma_1)^*(E) - \dfrac{a_2}{a_3 - a_1}\gamma_1^*(F) - \dfrac{1}{a_1}G_1, \\ S_{Y_1} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \gamma_1)^*\Big(-K_X\Big) - \dfrac{1}{a_3}(\beta \circ \gamma_1)^*(E) - \dfrac{1}{a_3 - a_1}\gamma_1^*(F) - \dfrac{1}{a_1}G_1, \\ E_{Y_1} \sim_{\mathbb{Q}} (\beta \circ \gamma_1)^*(E) - \dfrac{a_3 - 2a_1}{a_3 - a_1}\gamma_1^*(F) - \dfrac{1}{a_1}G_1, \\ F_{Y_1} \sim_{\mathbb{Q}} \gamma_1^*(F) - \dfrac{a_1 - 1}{a_1}G_1. \end{cases}$$

These equivalence shows that the intersection form of $\bar{C}_{Y_1}$ and L_{Y_1} on the surface B_{Y_1} is negative-definite.[6] Since $\mathcal{M}_{Y_1}|_{B_{Y_1}} \equiv n\bar{C}_{Y_1}+nL_{Y_1}$, the pencil $\mathcal{M}$ coincides with the linear system $|-a_2K_X|$ by Theorem 0.2.9. However, the linear system $|-a_2K_X|$ is not a pencil. $\square$

Lemma 3.3.4. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2, then $\mathcal{M} = |-a_1K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2. Let $\gamma_2 : Y_2 \to W$ be the Kawamata blow up at the point Q_2 with weights $(1, a_1, a_3 - 3a_1)$. Then, $|-a_1K_V|$ is the proper transform of the pencil $|-a_1K_X|$ and the base locus of $|-a_1K_V|$ consists of the irreducible curve C_V whose image to X is the base curve of the pencil $|-a_1K_X|$.

Let D be a general surface of the pencil $|-a_1K_V|$. Then, D is normal and we can consider the curve C_V as a divisor on D. We have $C_V^2 < 0$ but $\mathcal{M}_{Y_2}|_D \equiv nC_V$ by Lemma 0.2.6, which implies that $\mathcal{M}_{Y_2} = |-a_1K_V|$ by Theorem 0.2.9. $\square$

Proposition 3.3.5. *The linear systems $|-a_1K_X|$ and $\mathcal{P}$ is the only Halphen pencils on X.*

Proof. It immediately follows from the previous arguments. $\square$

3.4. Cases $\daleth = 57,\ 66,\ 81,$ and 86.

Suppose that $\daleth \in \{57, 66, 81, 86\}$. Then, the threefold $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$ always contains the point $O = (0:0:0:1:0)$. It is a singular point of X that is a quotient singularity of type $\frac{1}{a_3}(1, a_1, a_3 - a_1)$. The threefold X can be given by

$$t^{\frac{d-a_2}{a_3}} z + \sum_{i=0}^{\frac{d-a_2}{a_3}} t^i f_{d-ia_3}(x, y, z, w) = 0,$$

where f_i is a general quasihomogeneous polynomial of degree i.

We also have a commutative diagram as follows:

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point O with weights $(1, a_1, a_3 - a_1)$,
- η is an elliptic fibration.

Let $\mathcal{P}$ be the pencil defined by

$$\lambda x^{a_2} + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

We may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point O by Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7.

The exceptional divisor E of the birational morphism π contains two singular points P and Q that are quotient singularities of types $\frac{1}{a_3-a_1}(1, a_3 - a_2, a_2 - a_1)$ and $\frac{1}{a_1}(1, 2a_1 - a_3, a_3 - a_1)$, respectively.

Lemma 3.4.1. *If the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point P, then $\mathcal{M} = |-a_1K_X|$.*

Proof. The proof of Proposition 1.13.1 immediately implies that $\mathcal{M} = |-a_1K_X|$. $\square$

Lemma 3.4.2. *If the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q, then $\mathcal{M} = \mathcal{P}$.*

[6] The curves $\bar{C}_{Y_1}$, L_{Y_1}, and $B_{Y_1} \cap F$ are components of a fiber of the elliptic fibration $\eta \circ \gamma_1|_{B_{Y_1}}$, which implies that the intersection form of $\bar{C}_{Y_1}$ and L_{Y_1} on the surface B_{Y_1} is negative-definite.

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q. Let $\beta : W \to Y$ be the Kawamata blow up at the point Q with weights $(1, 2a_1 - a_3, a_3 - a_1)$ and let F be the exceptional divisor of the birational morphism β. Then,

$$\mathcal{P}_W \sim_{\mathbb{Q}} (\pi \circ \beta)^*\Big(- a_2 K_X\Big) - \frac{a_2}{a_3}\beta^*(E) - \frac{a_2}{a_1}F,$$

but the base locus of $\mathcal{P}_W$ consists of the irreducible curve $\bar{C}_W$ whose image to X is the unique base curve $\bar{C}$ of the linear system $\mathcal{P}$.

Let D and M be general surfaces in $\mathcal{P}_W$ and $\mathcal{M}_W$, respectively. Then, D is normal and

$$M\Big|_D \equiv -nK_W\Big|_D \equiv n\bar{C}_W,$$

but $\bar{C}_W^2 < 0$ on the surface D. Therefore, we obtain $\mathcal{M}_W = \mathcal{P}_W$ from Theorem 0.2.9. □

K3-Proposition 3.4.3. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. We use the same notation in the proof of Lemma 3.4.2. The exceptional divisor E is isomorphic to $\mathbb{P}(1, a_1, a_3 - a_1)$. The curve Δ defined by the intersection of a general surface in $\mathcal{P}_Y$ with E is a curve of degree a_2. Because $a_2 < a_3 + 1$, the curve Δ is a rational curve. The result follows from Theorem 0.2.10 since the curve Δ_W is not contained in the base locus of the pencil $\mathcal{P}_W$. □

Proposition 3.4.4. *The linear systems $|-a_1K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

Proof. Because we may assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains either the point P or the point Q, it immediately follows from Lemmas 3.4.1 and 3.4.2. □

3.5. Case $\gimel = 58$, hypersurface of degree 24 in $\mathbb{P}(1, 3, 4, 7, 10)$.

Let X be the hypersurface given by a general quasihomogeneous equation of degree 24 in $\mathbb{P}(1, 3, 4, 7, 10)$ with $-K_X^3 = \frac{1}{35}$. Then, the singularities of X consist of two singular points P and Q that are quotient singularities of types $\frac{1}{7}(1, 3, 4)$ and $\frac{1}{10}(1, 3, 7)$, respectively, and one point of type $\frac{1}{2}(1, 1, 1)$. Also, by the generality of the hypersurface, we may assume that the hypersurface X is defined by the equation

$$w^2z + wf_{14}(x, y, z, t) + f_{24}(x, y, z, t) = 0,$$

where $f_i(x, y, z, t)$ is a quasihomogeneous polynomial of degree i. Proposition 0.3.12 implies that the linear system $|-3K_X|$ is a Halphen pencil. Let $\mathcal{P}$ be the pencil on X given by the equations

$$\lambda x^4 + \mu z = 0,$$

where $(\lambda, \mu) \in \mathbb{P}^1$. We will see that the linear system $\mathcal{P}$ is another Halphen pencil on X (K3-Proposition 3.5.4).

We have the following commutative diagram:

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_O"'] \arrow[dr, "\gamma_P"] \arrow[drr, "\eta"] & & \\
 & U_{PQ} \arrow[dl, "\beta_Q"'] \arrow[dr, "\beta_P"] & & U_{QO} \arrow[dl, "\beta_O"] & \mathbb{P}(1,3,4), \\
U_P \arrow[dr, "\alpha_P"'] & & U_Q \arrow[dl, "\alpha_Q"] & & \\
 & X \arrow[uurrr, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1, 3, 4)$,

- α_Q is the Kawamata blow up at the point Q with weights $(1,3,7)$,
- β_Q is the Kawamata blow up with weights $(1,3,7)$ at the point whose image by the birational morphism α_P is the point Q,
- β_P is the Kawamata blow up with weights $(1,3,4)$ at the point whose image by the birational morphism α_Q is the point P,
- β_O is the Kawamata blow up with weights $(1,3,4)$ at the singular point O of the variety U_Q that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of the birational morphism α_Q,
- γ_P is the Kawamata blow up with weights $(1,3,4)$ at the point whose image by the birational morphism $\alpha_Q\circ\beta_O$ is the point P,
- γ_O is the Kawamata blow up with weights $(1,3,4)$ at the singular point of the variety U_{PQ} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of the birational morphism β_Q,
- η is an elliptic fibration.

Because of Lemma 0.3.3 and Corollary 0.3.7, we may assume that

$$\mathbb{CS}\Big(X,\frac{1}{n}\mathcal{M}\Big)\subset\Big\{P,Q\Big\}.$$

The exceptional divisor E_P of the birational morphism α_P contains two quotient singular points P_1 and P_2 of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively.

Lemma 3.5.1. *If the set $\mathbb{CS}(U_P,\frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1, then $\mathcal{M}=|-3K_X|$.*

Proof. Suppose it contains the point P_1. Let $\beta_1: W_1\to U_P$ be the Kawamata blow up at the point P_1 with weights $(1,3,1)$. Then, the pencil $|-3K_{W_1}|$ is the proper transform of the pencil system $|-3K_X|$. Its base locus consists of the irreducible curve C_{W_1} whose image to X is the base curve of the pencil $|-3K_X|$. Then, $-K_{W_1}\cdot C_{W_1}<0$ and $\mathcal{M}_{W_1}\sim_{\mathbb{Q}}-nK_{W_1}$ imply $\mathcal{M}=|-3K_X|$ by Theorem 0.2.9. □

Lemma 3.5.2. *The set $\mathbb{CS}(U_P,\frac{1}{n}\mathcal{M}_{U_P})$ cannot contain the point P_2.*

Proof. Suppose it contains the point P_2. Let $\beta_2: W_2\to U_P$ be the Kawamata blow up at the point P_2 with weights $(1,2,1)$. Also, let $\mathcal{D}_2$ be the proper transform of the linear system $|-4K_X|$ by the birational morphism $\alpha_P\circ\beta_2$. Its base locus consists of the irreducible curve $\bar{C}_{W_2}$ whose image to X is the base curve of the linear system $|-4K_X|$. A general surface D_2 in $\mathcal{D}_2$ is normal and the self-intersection $\bar{C}_{W_2}^2$ is negative on the surface D_2. Because $\mathcal{M}_{W_2}|_{D_2}\equiv-n\bar{C}_{W_2}$, we obtain an absurd identity $\mathcal{M}=|-4K_X|$ from Theorem 0.2.9. □

Meanwhile, the exceptional divisor $E\cong\mathbb{P}(1,3,7)$ of the birational morphism α_Q contains two singular points O and Q_1 of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{3}(1,2,1)$. For the convenience, let L be the unique curve contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)|$ on E .

Lemma 3.5.3. *If the set $\mathbb{CS}(U_Q,\frac{1}{n}\mathcal{M}_{U_Q})$ contains the point Q_1, then $\mathcal{M}=\mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(U_Q,\frac{1}{n}\mathcal{M}_{U_Q})$ contains the point Q_1. Let $\pi_1: V_1\to U_Q$ be the Kawamata blow up at the point Q_1 with weights $(1,2,1)$. The base locus of $\mathcal{P}_{V_1}$ consists of two irreducible curves $\bar{C}_{V_1}$ and L_{V_1}.

For a general surface D_{V_1} in $\mathcal{P}_{V_1}$, we have

$$S_{V_1}\cdot D_{V_1}=\bar{C}_{V_1}+L_{V_1},\ E_{V_1}\cdot D_{V_1}=4L_{V_1}.$$

Using the following equivalences

$$\begin{cases}E_{V_1}\sim_{\mathbb{Q}}\pi_1^*(E)-\dfrac{2}{3}F_Q,\\ S_{V_1}\sim_{\mathbb{Q}}\big(\alpha_Q\circ\pi_1\big)^*\Big(-K_X\Big)-\dfrac{1}{10}\pi_1^*(E)-\dfrac{1}{3}F_Q,\\ D_{V_1}\sim_{\mathbb{Q}}\big(\alpha_Q\circ\pi_1\big)^*\Big(-4K_X\Big)-\dfrac{4}{10}\pi_1^*(E)-\dfrac{4}{3}F_Q,\end{cases}$$

where F_Q is the exceptional divisor of π_1, we can obtain

$$D_{V_1}\cdot \bar{C}_{V_1} = D_{V_1}\cdot L_{V_1} = -\frac{8}{7}.$$

Because $\pi_1^*(-K_{U_Q})\cdot L_1 = \pi_1^*(-K_{U_Q})\cdot \bar{C}_{V_1} = \frac{1}{21}$, the divisor $B := 24\pi_1^*(-K_{U_Q}) + D_{V_1}$ is nef and big and $B\cdot \bar{C}_{V_1} = B\cdot L_{V_1} = 0$. Therefore, Theorem 0.2.9 implies $\mathcal{M} = \mathcal{P}$. □

K3-Proposition 3.5.4. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. We use the same notations in the proof of Lemma 3.5.3. The exceptional divisor F_Q is isomorphic to $\mathbb{P}(1,2,1)$. Then, the intersection $\Delta := F_Q\cdot D_{V_1}$ is a curve of degree 4 on $\mathbb{P}(1,2,1)$.

Easy calculation shows that the curve Δ does not pass through the singular point of the surface F_Q. We suppose that the curve Δ is smooth. Because it does not pass through any singular point of F_Q and its degree on F_Q is four, it is an elliptic curve. One can see that the singularities of the image surface $D_{U_Q} := \pi_1(D_{V_1})$ are rational except the point Q_1. Then, the same argument of K3-Proposition 3.2.3 gives a contradiction. Therefore, the surface D_{V_1} is birational to a smooth K3 surface. □

The exceptional divisor F_O of the birational morphism β_O contains two quotient singular points O_1 and O_2 of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively.

Lemma 3.5.5. *If the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the point O_1, then $\mathcal{M} = |-3K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the point O_1. Let $\sigma_1 : U_1 \to U_{QO}$ be the Kawamata blow up at the point O_1 with weights $(1,3,1)$. Then, the pencil $|-3K_{U_1}|$ is the proper transform of the pencil $|-3K_X|$. Its base locus consists of the irreducible curve C_{U_1}. Because we have $\mathcal{M}_{U_1} \sim_{\mathbb{Q}} -nK_{U_1}$ and $-K_{U_1}\cdot C_{U_1} < 0$, we obtain $\mathcal{M} = |-3K_X|$ from Theorem 0.2.9. □

Lemma 3.5.6. *The set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ cannot contain the point O_2.*

Proof. Suppose that the set $\mathbb{CS}(U_{QO}, \frac{1}{n}\mathcal{M}_{U_{QO}})$ contains the point O_2. Let $\sigma_2 : U_2 \to U_{QO}$ be the Kawamata blow up at the point O_2 with weights $(1,2,1)$.

The base locus of the pencil $\mathcal{P}_{U_2}$ consists of the irreducible curves $\bar{C}_{U_2}$ and L_{U_2}. For a general surface H in $\mathcal{P}_{U_2}$, we have

$$H \sim_{\mathbb{Q}} (\alpha_Q\circ\beta_O\circ\sigma_2)^*(-4K_X) - \frac{4}{10}(\beta_O\circ\sigma_2)^*(E) - \frac{4}{7}\sigma_2^*(F_O) - \frac{1}{3}G,$$

where G is the exceptional divisor of σ_2. The general surface H is normal, $S_{U_2}\cdot H = \bar{C}_{U_2} + L_{U_2}$, and $E_{U_2}\cdot H = 4L_{U_2}$. Since

$$E_{U_2} \sim_{\mathbb{Q}} (\beta_O\circ\sigma_2)^*(E) - \frac{4}{7}\sigma_2^*(F_O) - \frac{1}{3}G,$$

we can see that the intersection form of the curves L_{U_2} and $\bar{C}_{U_2}$ on the surface H is negative-definite. The equivalence $\mathcal{M}_{U_2}|_H \equiv n\bar{C}_{U_2} + nL_{U_2}$ holds. Therefore, we can obtain $\mathcal{M} = \mathcal{P}$ from Theorem 0.2.9. However, $H \not\sim_{\mathbb{Q}} -4K_{U_2}$. □

Proposition 3.5.7. *The linear systems $|-3K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

Proof. Due to the previous lemmas, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{P, Q\Big\}.$$

Following the Kawamata blow ups $Y \to U_{QO} \to U_Q \to X$ and using Lemmas 3.5.3, 3.5.5, and 3.5.6, we can furthermore assume that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains one of singular points contained the exceptional divisor of the birational morphism γ_P. In this case, Lemmas 3.5.1 and 3.5.2 imply the statement. □

3.6. Case $\daleth = 60$, hypersurface of degree 24 in $\mathbb{P}(1,4,5,6,9)$.

The threefold X is a general hypersurface of degree 24 in $\mathbb{P}(1,4,5,6,9)$ with $-K_X^3 = \frac{1}{45}$. Its singularities consist of one singular point O that is a quotient singularity of type $\frac{1}{9}(1,4,5)$, one quotient singular point of type $\frac{1}{5}(1,4,1)$, one quotient singular point of type $\frac{1}{3}(1,1,2)$, and two quotient singular points of type $\frac{1}{2}(1,1,1)$.

It cannot be birationally transformed into an elliptic fibration ([4]). However, a general fiber of the natural projection $\xi : X \dashrightarrow \mathbb{P}(1,4)$ is birational to a smooth K3 surface by Proposition 0.3.12, in other words, the linear system $|-4K_X|$ is a Halphen pencil.

By coordinate change, we may assume that the threefold X is given by the equation

$$w^2t + wf_{15}(x,y,z,t) + f_{24}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil consisting of surfaces cut out on the threefold X by the equations

$$\lambda x^6 + \mu t = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

K3-Proposition 3.6.1. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. It is a compactification of a double cover of $\mathbb{C}^2$ branched over a curve of degree 6. Therefore, the statement follows from Theorem 0.1.3. □

We are to show that the pencils $|-4K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{5}(1,4,1)$, then $\mathcal{M} = |-4K_X|$ by Lemma 0.3.11. Therefore, due to Lemma 0.3.3 and Corollary 0.3.7, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$.

Let $\pi : Y \to X$ be the Kawamata blow up at the point O with weights $(1,4,5)$ and E be its exceptional divisor. Then, $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ by Lemma 0.2.6. Thus, the singularities of the log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ are not terminal by Theorem 0.2.4 because the divisor $-K_Y$ is nef and big.

The exceptional divisor $E \cong \mathbb{P}(1,4,5)$ contains two quotient singular points P and Q that are singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{5}(1,4,1)$ on the threefold Y, respectively. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains either the singular point P or the singular point Q by Lemma 0.2.7.

Lemma 3.6.2. *If the log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ is not terminal at the point Q, then $\mathcal{M} = |-4K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains the point Q. Let $\alpha : U \to Y$ be the Kawamata blow up at the point Q with weights $(1,4,1)$. Then, $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ by Lemma 0.2.6. The linear system $|-4K_U|$ is the proper transform of the pencil $|-4K_X|$. Its base locus consists of the irreducible curve C_U. Since

$$-K_U \cdot C_U = -4K_U^3 = -\frac{4}{30}$$

we obtain the identity $\mathcal{M} = |-4K_X|$ from Theorem 0.2.9. □

We may assume that $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y) = \{P\}$. Let $\beta : W \to Y$ be the Kawamata blow up at the singular point P with weights $(1,3,1)$ and F be its exceptional divisor. Then,

$$\mathcal{M}_W \sim_{\mathbb{Q}} (\pi\circ\beta)^*\Big(-nK_X\Big) - \frac{n}{9}\beta^*(E) - \frac{n}{4}F \sim_{\mathbb{Q}} -nK_W$$

by Lemma 0.2.6. In addition, we see

$$\mathcal{P}_W \sim_{\mathbb{Q}} (\pi\circ\beta)^*\Big(-6K_X\Big) - \frac{6}{9}\beta^*(E) - \frac{6}{4}F \sim_{\mathbb{Q}} -6K_W.$$

Let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,4,9)}(1)|$ on the surface E. The base locus of $\mathcal{P}_W$ consists of the irreducible curve $\tilde{C}_W$ and the irreducible curve L_W.

A general surface D_W in $\mathcal{P}_W$ is normal. From the equivalences

$$\begin{cases} E_W \sim_{\mathbb{Q}} \beta^*\big(E\big) - \dfrac{3}{4}F, \\ S_W \sim_{\mathbb{Q}} \big(\pi\circ\beta\big)^*\Big(-K_X\Big) - \dfrac{1}{9}\beta^*\big(E\big) - \dfrac{1}{4}F, \\ D_W \sim_{\mathbb{Q}} \big(\pi\circ\beta\big)^*\Big(-6K_X\Big) - \dfrac{6}{9}\beta^*\big(E\big) - \dfrac{6}{4}F, \end{cases}$$

we obtain

$$L_W^2 = \tilde{C}_W^2 = -\frac{1}{5},\ L_W\cdot\tilde{C}_W = 0$$

on the surface D_W because $S_W\cdot D_W = \tilde{C}_W + L_W$ and $D_W\cdot E_W = 6L_W$.

For a general surface M_W of the linear system $\mathcal{M}_W$,

$$M_W\Big|_{D_W} \equiv -nK_W\Big|_{D_W} \equiv nS_W\Big|_{D_W} \equiv n\tilde{C}_W + nL_W,$$

but the intersection form of the irreducible curves L_W and $\tilde{C}_W$ on the surface D_W is negative-definite. Therefore, Theorem 0.2.9 implies that $\mathcal{M}_W = \mathcal{P}_W$.

Consequently, we have obtained

Proposition 3.6.3. *The linear systems $|-4K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

3.7. Case $\daleth = 69$, hypersurface of degree 28 in $\mathbb{P}(1,4,6,7,11)$.

The variety X is a general hypersurface of degree 28 in $\mathbb{P}(1,4,6,7,11)$ with $-K_X^3 = \frac{1}{66}$. The singularities of the hypersurface X consist of one singular point O that are quotient singularities of type $\frac{1}{11}(1,4,7)$, one point of type $\frac{1}{6}(1,1,5)$, and two points of type $\frac{1}{2}(1,1,1)$.

There is a commutative diagram

$$\begin{tikzcd} U & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[u, leftarrow, "\alpha"'] \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,3,4), \end{tikzcd}$$

where

- α is the Kawamata blow up at the point O with weights $(1,4,7)$,
- β is the Kawamata blow up with weights $(1,4,3)$ at the singular point of the variety U contained in the exceptional divisor of the birational morphism α that is a quotient singularity of type $\frac{1}{7}(1,4,3)$,
- η is an elliptic fibration.

The linear system $|-4K_X|$ is a Halphen pencil on X by Proposition 0.3.12. However, the pencil $|-4K_X|$ is not a unique Halphen pencil on X. Indeed, the hypersurface X can be given by equation

$$w^2z + wf_{17}(x,y,z,t) + f_{28}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces cut out on the hypersurface X by the equations

$$\lambda x^6 + \mu z = 0,$$

where $(\lambda:\mu)\in\mathbb{P}^1$. We will see that the linear system $\mathcal{P}$ is another Halphen pencil.

First of all, if the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{6}(1,1,5)$, the identity $\mathcal{M} = |-4K_X|$ follows from Lemma 0.3.11. Moreover, due to Lemmas 0.3.3, 0.3.10, and Corollary 0.3.8, we may assume that $\mathbb{CS}(X,\frac{1}{n}\mathcal{M}) = \{O\}$.

It follows from Theorem 0.2.4 that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ is not empty.

The exceptional divisor $E\cong\mathbb{P}(1,4,7)$ of the birational morphism α contains two singular point P and Q of U that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{4}(1,1,3)$, respectively.

For the convenience, let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,4,7)}(1)|$ on the surface E.

Lemma 3.7.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ does not contain the point Q.*

Proof. Suppose that $Q \in \mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$. Let $\pi_Q : U_Q \to U$ be the Kawamata blow up at the point Q with weights $(1, 1, 3)$.

Let $\mathcal{D}_{U_Q}$ be the proper transform of the linear system $|-7K_X|$ by the birational morphism $\alpha \circ \pi_Q$. Its base locus consists of the irreducible curve $\tilde{C}_{U_Q}$. A general surface D_{U_Q} in $\mathcal{D}_{U_Q}$ is normal. Moreover, the inequality $\tilde{C}_{U_Q}^2 < 0$ holds on the surface D_{U_Q}. It implies the identity $\mathcal{M}_{U_Q} = \mathcal{D}_{U_Q}$ by Theorem 0.2.9 because $\mathcal{M}_{U_Q}|_{D_{U_Q}} \equiv n\tilde{C}_{U_Q}$. However, the linear system $\mathcal{D}_{U_Q}$ is not a pencil, which is a contradiction. □

Therefore, by Theorem 0.2.4 and Lemma 0.2.7, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point P.

The exceptional divisor F of the birational morphism β contains two singular points P_1 and P_2 of W that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 3, 1)$, respectively.

As usual, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must contain either the point P_1 or the point P_2 by Theorem 0.2.4 and Lemma 0.2.7.

Lemma 3.7.2. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_1, then $\mathcal{M} = |-4K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_1. Let $\pi_1 : W_1 \to W$ be the Kawamata blow up at the point P_1 with weights $(1, 1, 2)$. Then, $|-4K_{W_1}|$ is the proper transform of the pencil $|-4K_X|$. Its base locus consists of the irreducible curve C_{W_1}. Because $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$, the inequality $-K_{W_1} \cdot C_{W_1} < 0$ implies the identity $\mathcal{M} = |-4K_X|$ by Theorem 0.2.9. □

Lemma 3.7.3. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_2, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point P_2. Let $\pi_2 : W_2 \to W$ be the Kawamata blow up at the point P_2 with weights $(1, 3, 1)$ and let G be its exceptional divisor. Then, the base locus of the pencil $\mathcal{P}_{W_2}$ consists of the irreducible curves $\bar{C}_{W_2}$ and L_{W_2}.

Let D be a general surface of the pencil $\mathcal{P}$. Then, $D_{W_2} \sim_{\mathbb{Q}} -6K_{W_2}$, while $S_{W_2} \cdot D_{W_2} = 2L_{W_2} + \bar{C}_{W_2}$ and $E_{W_2} \cdot D_{W_2} = 6L_{W_2}$.

Let us find a divisor B on the threefold W_2 such that B is nef and big but $R \cdot C_{W_2} = R \cdot L_{W_2} = 0$. Namely, consider a divisor B such that

$$B = (\alpha \circ \beta \circ \pi_2)^*(-\lambda K_X) + (\beta \circ \pi_2)^*(-\mu K_U) + D_{W_2},$$

where λ and μ are nonnegative rational numbers with $(\lambda, \mu) \neq (0, 0)$. Then, the equalities $B \cdot \bar{C}_{W_2} = B \cdot L_{W_2} = 0$ imply that $\lambda = 0$ and $\mu = 42$ because

$$-K_X \cdot \alpha \circ \beta \circ \pi_2(\bar{C}_{W_2}) = \frac{1}{11},\ -K_U \cdot \beta \circ \pi_2(L_{W_2}) = \frac{1}{28},\ -K_U \cdot \beta \circ \pi_2(\bar{C}_{W_2}) = 0,$$

$$D_{W_2} \cdot L_{W_2} = -\frac{3}{2},\ D_{W_2} \cdot \bar{C}_{W_2} = 0.$$

The divisor B is nef and big because the divisors $-K_X$ and $-K_U$ are nef and big, while $\bar{C}_{W_2}$ and L_{W_2} are the only curves on the threefold W_2 that have negative intersection with D_{W_2}. Let M be a general surface in $\mathcal{M}_{W_2}$. Then, $B \cdot D_{W_2} \cdot M = 0$, which implies that $\mathcal{M}_{W_2} = \mathcal{P}_{W_2}$ by Theorem 0.2.9. □

Remark 3.7.4. The surface D_{W_2} is not normal. Indeed, it follows from local computations that the surface D_{W_2} is singular along the curve C_{W_2}, which is reflected by the fact that $S_{W_2} \cdot D_{W_2} = 2L_{W_2} + C_{W_2}$.

K3-Proposition 3.7.5. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. Let Δ be the curve on the exceptional divisor F of the birational morphism β defined by intersecting with a general surface in $\mathcal{P}_W$. Then, the curve Δ is a curve of degree 6 in the surface $\mathbb{P}(1,4,3)$, and hence it is rational. The proper transform Δ_{W_2} is a rational curve not contained in the base locus of the pencil $\mathcal{P}_{W_2}$. Therefore, a general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface by Corollary 0.2.12. □

Consequently, we have shown

Proposition 3.7.6. *The linear systems $|-4K_X|$ and $\mathcal{P}$ are the only pencils on X.*

3.8. Case $\daleth = 74$, hypersurface of degree 30 in $\mathbb{P}(1,3,4,10,13)$.

The hypersurface X is defined by a general quasihomogeneous equation of degree 30 in $\mathbb{P}(1,3,4,10,13)$ with $-K_X^3 = \frac{1}{52}$. Its singularities consist of one quotient singularity O of type $\frac{1}{13}(1,3,10)$, one quotient singular point of type $\frac{1}{4}(1,3,1)$, and one quotient singular point of type $\frac{1}{2}(1,1,1)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,4), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,3,10)$,
- β is the Kawamata blow up with weights $(1,3,7)$ at the singular point of the variety U that is a quotient singularity of type $\frac{1}{10}(1,3,7)$ contained in the exceptional divisor of the birational morphism α,
- γ is the Kawamata blow up with weights $(1,3,4)$ at the singular point of the variety W that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of the birational morphism β,
- η is an elliptic fibration.

Proposition 0.3.12 implies that the linear system $|-3K_X|$ is a Halphen pencil. There is another Halphen pencil as follows: The hypersurface X can be given by the equation

$$w^2z + wf_{17}(x,y,z,t) + f_{30}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil on X given by the equations

$$\lambda x^4 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$. Then, the linear system $\mathcal{P}$ is another Halphen pencil on X (K3-Proposition 3.8.2).

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the singular point of type $\frac{1}{4}(1,3,1)$, then the identity $\mathcal{M} = |-3K_X|$ follows from Lemma 0.3.11. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}$$

due to Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, and Corollary 0.3.7.

The exceptional divisor $E \cong \mathbb{P}(1,3,10)$ of the birational morphism α contains two quotient singular points P_1 and Q_1 of types $\frac{1}{10}(1,3,7)$ and $\frac{1}{3}(1,2,1)$, respectively. For the convenience, let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,10)}(1)|$ on the surface E.

Lemma 3.8.1. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q_1, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point Q_1. Let $\alpha_1 : W_1 \to U$ be the Kawamata blow up at the point Q_1 with weights $(1, 2, 1)$. By Lemma 0.2.6, we have $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -K_{W_1}$.

The base locus of the pencil $\mathcal{P}_{W_1}$ consists of the curves $\bar{C}_{W_1}$ and L_{W_1}.

Let D_{W_1} be a general surface in $\mathcal{P}_{W_1}$. We see then that

$$S_{W_1} \cdot D_{W_1} = \bar{C}_{W_1} + 2L_{W_1}, \quad E_{W_1} \cdot D_{W_1} = 4L_{W_1}.$$

Note that the surface D_{W_1} is not normal and

$$\begin{cases} D_{W_1} \sim_{\mathbb{Q}} (\alpha \circ \alpha_1)^*(-4K_X) - \dfrac{4}{13}\alpha_1^*(E) - \dfrac{4}{3}E_1, \\ S_{W_1} \sim_{\mathbb{Q}} (\alpha \circ \alpha_1)^*(-K_X) - \dfrac{1}{13}\alpha_1^*(E) - \dfrac{1}{3}E_1, \\ E_{W_1} \sim_{\mathbb{Q}} \alpha_1^*(E) - \dfrac{2}{3}E_1, \end{cases}$$

where E_1 is the exceptional divisor of the birational morphism α_1.

One can easily see that

$$-K_U \cdot L = \frac{1}{30}, \quad -K_U \cdot \bar{C}_U = 0, \quad D_{W_1} \cdot L_{W_1} = -\frac{6}{5}, \quad D_{W_1} \cdot \bar{C}_{W_1} = 0.$$

Because $-K_U$ is nef and big and L_{W_1} is the only curve intersecting D_{W_1} negatively, $B := 36\alpha_1^*(-K_U) + D_{W_1}$ is also a nef and big divisor with $B \cdot L_{W_1} = B \cdot \bar{C}_{W_1} = 0$. Let M be a general surface in $\mathcal{M}_{W_1}$. We then obtain $B \cdot M \cdot D_{W_1} = 0$, which implies that $\mathcal{M} = \mathcal{P}$. $\square$

K3-Proposition 3.8.2. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. We use the same notations in the proof of Lemma 3.8.1. Let Δ be the curve on the exceptional divisor E_1 defined by intersecting with the surface D_{W_1}. Then, the curve Δ is a curve of degree 4 on the surface $\mathbb{P}(1, 2, 1)$.

If the curve Δ is singular, then it is a rational curve on the surface D_{W_1}. Suppose that the curve Δ is smooth. Then, it cannot pass through the singular point of the surface E_1, and hence it is an elliptic curve. Then, the argument in the proof of K3-Proposition 3.2.3 gives a contradiction. Therefore, the surface D_{W_1} must have a rational curve not contained in the base locus of the pencil $\mathcal{P}_{W_1}$. Then, Corollaries 0.2.11 and 0.2.12 complete the proof. $\square$

Due to Lemma 3.8.1, we may assume that $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) = \{P_1\}$.

The exceptional divisor $F \cong \mathbb{P}(1, 3, 7)$ of the birational morphism β contains two singular points P_2 and Q_2 that are quotient singularities of types $\frac{1}{7}(1, 3, 4)$ and $\frac{1}{3}(1, 2, 1)$, respectively. We let $\bar{L}$ be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)|$ on surface F.

The set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains either the point P_2 or the point Q_2.

Lemma 3.8.3. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_2. Let $\beta_2 : W_2 \to W$ be the Kawamata blow up at the point Q_2 with weights $(1, 2, 1)$ and F_2 be its exceptional divisor. The base locus of the pencil $\mathcal{P}_{W_2}$ consists of three irreducible curves $\bar{C}_{W_2}$, L_{W_2}, and $\bar{L}_{W_2}$.

Let D_{W_2} be a general surface in the pencil $\mathcal{P}_{W_2}$. Then, we have

$$\begin{cases} D_{W_2} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \beta_2)^*(-4K_X) - \dfrac{4}{13}(\beta \circ \beta_2)^*(E) - \dfrac{4}{10}\beta_2^*(F) - \dfrac{4}{3}F_2, \\ S_{W_2} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \beta_2)^*(-K_X) - \dfrac{1}{9}(\beta \circ \beta_2)^*(E) - \dfrac{1}{7}\beta_2^*(F) - \dfrac{1}{3}F_2, \\ E_{W_2} \sim_{\mathbb{Q}} (\beta \circ \beta_2)^*(E) - \dfrac{7}{10}\beta_2^*(F) - \dfrac{1}{3}F_2, \\ F_{W_2} \sim_{\mathbb{Q}} \beta_2^*(F) - \dfrac{2}{3}F_2. \end{cases}$$

Furthermore,

$$S_{W_2}\cdot D_{W_2}=\bar{C}_{W_2}+2L_{W_2}+\bar{L}_{W_2},\quad E_{W_2}\cdot D_{W_2}=4L_{W_2},\quad F_{W_2}\cdot D_{W_2}=4\bar{L}_{W_2}.$$

First of all, we can easily check that

$$D_{W_2}\cdot L_{W_2}=-\frac{2}{3},\quad D_{W_2}\cdot \bar{L}_{W_2}=-\frac{8}{7},\quad D_{W_2}\cdot \bar{C}_{W_2}=0.$$

We then consider the divisor

$$B_2=20(\beta\circ\beta_2)^*(-K_U)+24\beta_2^*(-K_W)+D_{W_2}$$

on W_2. Because

$$\begin{cases} -K_U\cdot(\beta\circ\beta_2)(L_{W_2})=\dfrac{1}{30},\ -K_U\cdot(\beta\circ\beta_2)(\bar{C}_{W_2})=0\\ -K_W\cdot\beta_2(L_{W_2})=0,\ -K_W\cdot\beta_2(\bar{L}_{W_2})=\dfrac{1}{21},\ -K_W\cdot\beta_2(\bar{C}_{W_2})=0\end{cases}$$

we see that $B_2\cdot L_{W_2}=B_2\cdot\bar{L}_{W_2}=B_2\cdot\bar{C}_{W_2}=0$.

Because $-K_U$ and $-K_W$ are nef and big and L_{W_2} and $\bar{L}_{W_2}$ are the only curves intersecting D_{W_2} negatively, the divisor B_2 is also nef and big. Let M_2 be a general surface in $\mathcal{M}_{W_2}$. We then obtain $B_2\cdot M_2\cdot D_{W_2}=0$, which implies that $\mathcal{M}=\mathcal{P}$. □

Due to the lemma above, we may assume that $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)=\{P_2\}$. The exceptional divisor $G\cong\mathbb{P}(1,3,4)$ of the birational morphism γ contains two singular points P_3 and Q_3 that are quotient singularities of types $\frac{1}{4}(1,3,1)$ and $\frac{1}{3}(1,2,1)$, respectively. Again, the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains either the point P_3 or the point Q_3.

Lemma 3.8.4. *The set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ cannot contain the point Q_3.*

Proof. Suppose that the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains the point Q_3. Let $\gamma_3: W_3\to Y$ be the Kawamata blow up at the point Q_3 with weights $(1,2,1)$ and let G_3 be the exceptional divisor of γ_3. The base locus of the pencil $\mathcal{P}_{W_3}$ consists of three irreducible curves $\bar{C}_{W_3}$, L_{W_3}, and $\bar{L}_{W_3}$. Let D_{W_3} be a general surface in the pencil $\mathcal{P}_{W_3}$. We see

$$\begin{cases} D_{W_3}\sim_{\mathbb{Q}}(\alpha\circ\beta\circ\gamma\circ\gamma_3)^*(-4K_X)-\dfrac{4}{13}(\beta\circ\gamma\circ\gamma_3)^*(E)-\dfrac{4}{10}(\gamma\circ\gamma_3)^*(F)-\dfrac{4}{7}\gamma_3^*(G)-\dfrac{1}{3}G_3\\ S_{W_3}\sim_{\mathbb{Q}}(\alpha\circ\beta\circ\gamma\circ\gamma_3)^*(-K_X)-\dfrac{1}{13}(\beta\circ\gamma\circ\gamma_3)^*(E)-\dfrac{1}{10}(\gamma\circ\gamma_3)^*(F)-\dfrac{1}{7}\gamma_3^*(G)-\dfrac{1}{3}G_3,\\ E_{W_3}\sim_{\mathbb{Q}}(\beta\circ\gamma\circ\gamma_3)^*(E)-\dfrac{7}{10}(\gamma\circ\gamma_3)^*(F),\\ F_{W_3}\sim_{\mathbb{Q}}(\gamma\circ\gamma_3)^*(F)-\dfrac{4}{7}\gamma_3^*(G)-\dfrac{1}{3}G_3,\end{cases}$$

Furthermore,

$$S_{W_3}\cdot D_{W_3}=\bar{C}_{W_3}+2L_{W_3}+\bar{L}_{W_3},\quad E_{W_3}\cdot D_{W_3}=4L_{W_3},\quad F_{W_3}\cdot D_{W_3}=4\bar{L}_{W_3}.$$

From this, one can obtain

$$D_{W_3}\cdot L_{W_3}=0,\quad D_{W_3}\cdot\bar{L}_{W_3}=-\frac{1}{24},\quad D_{W_3}\cdot\bar{C}_{W_3}=-\frac{1}{8}.$$

Using the same method as in the previous lemma with nef and big divisors $-K_X$, $-K_U$, and $-K_W$, we can find a nef and big divisor B_3 on W_3 such that $B_3\cdot D_{W_3}\cdot M_3=0$, where M_3 is a general surface in $\mathcal{M}_{W_3}$, which implies that $\mathcal{M}=\mathcal{P}$. However it is a contradiction because $D_3\not\sim_{\mathbb{Q}}-4K_{W_3}$. □

Therefore, the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains the point P_3. Let $\delta: V\to Y$ be the Kawamata blow up at the point P_3 with weights $(1,3,1)$. The pencil $|-3K_V|$ is the proper transform of the

pencil $|-3K_X|$. It has only one irreducible base curve C_V. For a general surface M in $\mathcal{M}_V$, we have

$$M\Big|_{D_V} \equiv -nK_V\Big|_{D_V} \equiv nC_V.$$

Therefore, the inequality $-K_C \cdot C_V < 0$ implies that $\mathcal{M} = |-3K_X|$ by Theorem 0.2.9.

Consequently, we have proved

Proposition 3.8.5. *The linear systems $|-3K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

3.9. Case $\gimel = 76$, hypersurface of degree 30 in $\mathbb{P}(1,5,6,8,11)$.

The variety X is a general hypersurface of degree 25 in $\mathbb{P}(1,5,6,8,11)$ with $-K_X^3 = \frac{1}{88}$. The singularities of X consist of one point that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, one point P that is a quotient singularity of type $\frac{1}{8}(1,5,3)$, and one point Q that is a quotient singularity of type $\frac{1}{11}(1,5,6)$.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[ld, "\beta_Q"'] \arrow[rd, "\beta_P"] \arrow[rrrd, "\eta"] & & & \\
U_P \arrow[rd, "\alpha_P"'] & & U_Q \arrow[ld, "\alpha_Q"] & & \mathbb{P}(1,5,6), \\
 & X \arrow[rrru, dashed, "\psi"'] & & &
\end{tikzcd}$$

where

- ψ is the natural projection,
- α_P is the Kawamata blow up at the point P with weights $(1,5,3)$,
- α_Q is the Kawamata blow up at the point Q with weights $(1,5,6)$,
- β_Q is the Kawamata blow up with weights $(1,5,6)$ at the point whose image to X is the point Q,
- β_P is the Kawamata blow up with weights $(1,5,3)$ at the point whose image to X is the point P,
- η is an elliptic fibration.

The threefold X can be given by the equation

$$t^3z + t^2 f_{14}(x,y,z,w) + t\Big(w^2 + f_{22}(x,y,z)\Big) + f_{30}(x,y,z,w) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil consisting of surfaces cut out on the threefold X by the equations

$$\lambda x^6 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

Proposition 3.9.1. *The linear systems $|-5K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

By Lemmas 0.3.10, 0.3.3, and Corollary 0.3.7, we have $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) \subset \{P, Q\}$.

The exceptional divisor E_P of the birational morphism α_p contains two quotient singular points P_1 and P_2 of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{3}(1,2,1)$, respectively. The divisor $-K_{U_P}$ is nef and big. Thus, the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ is not empty by Theorem 0.2.4.

Lemma 3.9.2. *If the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains P_1, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains the point P_1. Let $\beta_1 : W_1 \to U_P$ be the Kawamata blow up at the point P_1 with weights $(1,2,3)$. Then, $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ by Lemma 0.2.6. Also, one can see that $\mathcal{P}_{W_1} \sim_{\mathbb{Q}} -6K_{W_1}$ and the base locus of $\mathcal{P}_{W_1}$ consists of the irreducible curve $\bar{C}_{W_1}$. The inequality $-K_{W_1} \cdot \bar{C}_{W_1} < 0$ implies $\mathcal{M} = \mathcal{P}$ by Theorem 0.2.9. $\square$

K3-Proposition 3.9.3. *A general surface in the pencil $\mathcal{P}$ is birational to a K3 surface.*

Proof. We use the same notation in the proof of Lemma 3.9.2. The exceptional divisor E_P is isomorphic to the weighted projective space $\mathbb{P}(1,3,5)$ and the curve Δ defined by the intersection of E_P with a general member in $\mathcal{P}_{U_P}$ has degree 6 on E_P. Therefore, the curve Δ_{W_1} is a rational curve not contained in the base locus of the pencil $\mathcal{P}_{W_1}$, and hence a general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface by Corollary 0.2.12. □

Lemma 3.9.4. *If the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains P_2, then $\mathcal{M} = |-5K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_P)$ contains the point P_2. Let $\beta_2 : W_2 \to U_P$ be the Kawamata blow up of the point P_2 with weights $(1,2,1)$. Then, $\mathcal{M}_{W_2} \sim_{\mathbb{Q}} -nK_{W_2}$ by Lemma 0.2.6. The pencil $|-5K_{W_2}|$ is the proper transform of the pencil $|-5K_X|$ and its base locus consists of the irreducible curve C_{W_2}. Then, the inequality $-K_{W_2} \cdot C_{W_2} < 0$ implies $\mathcal{M} = |-5K_X|$ by Theorem 0.2.9. □

Meanwhile, the exceptional divisor E_Q of the birational morphism α_Q contains two quotient singular points Q_1 and Q_2 of types $\frac{1}{5}(1,4,1)$ and $\frac{1}{6}(1,5,1)$, respectively. The divisor $-K_{U_Q}$ is nef and big. Thus, the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ is not empty by Theorem 0.2.4.

Lemma 3.9.5. *If the set $\mathbb{CS}(U_Q, \frac{1}{n}\mathcal{M}_{U_Q})$ contains Q_1, then $\mathcal{M} = \mathcal{P}$.*

Proof. The proof is similar to that of Lemma 3.9.2. □

Lemma 3.9.6. *If the set $\mathbb{CS}(U_P, \frac{1}{n}\mathcal{M}_{U_P})$ contains Q_2, then $\mathcal{M} = |-5K_X|$.*

Proof. The proof is similar to that of Lemma 3.9.4. □

Therefore, for the proof of Proposition 3.9.1, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P, Q\}$ by the previous lemmas.

Each member of the pencil $\mathcal{M}_Y$ is contracted to a curve by the elliptic fibration η by Lemma 0.2.6 but the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ is not empty by Theorem 0.2.4. Hence, it follows from Lemma 0.2.7 that the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ contains a singular point of the threefold Y that is contained either in the exceptional divisor of the birational morphism α_P or in the exceptional divisor of the birational morphism β_Q. Then, Lemmas 3.9.2, 3.9.4, 3.9.5, and 3.9.6 conclude the proof of Proposition 3.9.1.

3.10. Case $\daleth = 79$, hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$.

The threefold X is a general hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$ with $-K_X^3 = \frac{1}{70}$. It has two singular points. One is a quotient singularity of type $\frac{1}{5}(1,1,4)$ and the other is a quotient singularity O of type $\frac{1}{14}(1,3,11)$. The hypersurface X can be given by the equation

$$w^2z + wf_{19}(x,y,z,t) + f_{33}(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil cut out on X by

$$\lambda x^5 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\ X \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}(1,3,5), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point O with weights $(1,3,11)$,

- β is the Kawamata blow up with weights $(1,3,8)$ at the singular point of type $\frac{1}{11}(1,3,8)$ contained in the exceptional divisor of the birational morphism α,
- γ is the Kawamata blow up with weights $(1,3,5)$ at the singular point of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of the birational morphism β,
- η is an elliptic fibration.

If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}_X)$ contains the singular point of type $\frac{1}{5}(1,1,4)$, then $\mathcal{M} = |-3K_X|$ by Lemma 0.3.11. Therefore, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) = \Big\{O\Big\}$$

due to Lemma 0.3.3 and Corollary 0.3.7.

The exceptional divisor E of α contains two quotient singular points P and Q of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{11}(1,3,8)$, respectively.

Lemma 3.10.1. *The set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q.*

Proof. Suppose that $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U) \neq \{Q\}$. Then, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point P. Let $\pi_P : U_P \to U$ be the Kawamata blow up at P with weights $(1,1,2)$. Then, $\mathcal{M}_{U_P} \sim_{\mathbb{Q}} -nK_{U_P}$ by Lemma 0.2.6.

Let $\mathcal{D}$ be the proper transforms of $|-11K_X|$ on the threefold U_P and D be a general surface of the linear system $\mathcal{D}$. Then, the base locus of the linear system $\mathcal{D}$ does not contain curves, which implies that the divisor D is nef. Thus, we obtain an absurd inequality

$$0 \leq D \cdot M_1 \cdot M_2 = -\frac{n^2}{5},$$

where M_1 and M_2 are general surfaces of the pencil $\mathcal{M}_{U_P}$. □

The exceptional divisor F of the birational morphism β contains two singular points Q_1 and Q_2 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{8}(1,3,5)$ respectively.

Lemma 3.10.2. *If the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1, then $\mathcal{M} = \mathcal{P}$.*

Proof. Suppose that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1. Let $\pi : W_1 \to W$ be the Kawamata blow up of Q_1 with weights $(1,1,2)$ and G be its exceptional divisor. Then, $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ by Lemma 0.2.6.

Let $\mathcal{L}$ be the linear system on the hypersurface X cut out by

$$\lambda_0 x^{30} + \lambda_1 y^{10} + \lambda_2 z^6 + \lambda_3 t^2 x^8 + \lambda_4 t^2 y^2 x^2 + \lambda_5 t y^6 x + \lambda_6 wtz = 0$$

where $(\lambda_0 : \cdots : \lambda_6) \in \mathbb{P}^6$. Then, the base locus of $\mathcal{L}$ does not contain curves. Then, it follows from simple calculations that the base locus of the linear system $\mathcal{L}_{W_1}$ does not contain any curve and for a general surface B in $\mathcal{L}$, we obtain

$$B_{W_1} \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*\Big(-30K_X\Big) - \frac{30}{14}(\beta \circ \pi)^*(E) - \frac{8}{11}\pi^*(F) - \frac{2}{3}G.$$

In particular, the divisor B_{W_1} is nef and big.

Let M be a general surface of the pencil $\mathcal{M}_{W_1}$ and D be a general surface of the linear system $|-5K_{W_1}|$. Then, $B_{W_1} \cdot M \cdot D = 0$, which implies that $\mathcal{M} = \mathcal{P}$ by Theorem 0.2.9 because the linear system $|-5K_{W_1}|$ is the proper transform of the pencil $\mathcal{P}$. □

K3-Proposition 3.10.3. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. We use the same notations in the proof of Lemma 3.10.2. The surface G is isomorphic to the projective space $\mathbb{P}(1,1,2)$. Let T be a general surface in the pencil $\mathcal{P}$ and let $\Delta = G \cdot T_{W_1}$. Then, by simple calculation, we see that the curve Δ on G is defined by the equation

$$\epsilon_1 \bar{x}^5 + \epsilon_2 \bar{x}^2 \bar{y}\bar{t} + \epsilon_3 \bar{x}\bar{y}^2\bar{t} + \epsilon_4 \bar{y}\bar{t}^2 = 0 \subset \mathrm{Proj}(\mathbb{C}[\bar{x}, \bar{y}, \bar{t}]) = \mathbb{P}(1,1,2),$$

where each ϵ_i is a general complex number. It has two nodes at the points $(1:0:0)$ and $(0:1:0)$. But it is smooth at the point $(0:0:1)$ which is a $\mathbb{A}_1$ singular point of the surface G.

Let $\tilde{G}$ be the blow up of the surface G at these three points. The genus of the normalization $\tilde{\Delta}$ of the curve Δ is

$$p_g(\tilde{\Delta}) = \frac{(K_{\tilde{G}} + \tilde{\Delta}) \cdot \tilde{\Delta}}{2} + 1 = \frac{(K_G + \Delta) \cdot \Delta - \frac{1}{2}}{2} + 1 - 2 = 0,$$

and hence the curve Δ is a rational curve not contained in the base locus of the pencil $\mathcal{P}_{W_1}$. Therefore, Corollary 0.2.12 completes the proof. □

We may assume that $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W) = \{Q_2\}$ due to Theorem 0.2.4 and Lemma 0.2.7. Let O_1 and O_2 be the quotient singular points of the threefold Y contained in the exceptional divisor of γ that are of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{5}(1,3,2)$, respectively. Then,

$$\varnothing \neq \mathbb{CS}\Big(Y, \frac{1}{n}\mathcal{M}_Y\Big) \subset \Big\{O_1, O_2\Big\}$$

by Theorem 0.2.4, Lemmas 0.2.6, and 0.2.7. The proof of Lemma 3.3.3 implies that the $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain the point O_1. Now, the proofs of Lemma 3.3.4 shows $\mathcal{M} = |-3K_X|$.

Therefore, we have proved

Proposition 3.10.4. *The linear systems $|-3K_X|$ and $\mathcal{P}$ are the only Halphen pencils on X.*

3.11. Cases $\daleth = 84$ and 93.

In the case of $\daleth = 84$, the threefold X is a general hypersurface of degree 36 in $\mathbb{P}(1,7,8,9,12)$ with $-K_X^3 = \frac{1}{168}$. Its singularities consist of one quotient singular point of type $\frac{1}{4}(1,3,1)$, one quotient singular point of type $\frac{1}{3}(1,1,2)$, one quotient singular point of type $\frac{1}{7}(1,2,5)$, and one quotient singular point of type $\frac{1}{8}(1,7,1)$.

In the case of $\daleth = 93$, the threefold X is a general hypersurface of degree 50 in $\mathbb{P}(1,7,8,10,25)$ with $-K_X^3 = \frac{1}{280}$. It has one quotient singular point of type $\frac{1}{2}(1,1,1)$, one quotient singular point of type $\frac{1}{5}(1,2,3)$, one quotient singular point of type $\frac{1}{7}(1,3,4)$, and one quotient singular point of type $\frac{1}{8}(1,7,1)$.

In both cases, the threefold X cannot be birationally transformed to an elliptic fibration ([4]). However, it can be rationally fibred by K3 surfaces.

The threefold X can be given by the equation

$$y^{\frac{d-8}{7}}z + \sum_{i=0}^{\frac{d-15}{7}} y^i f_{d-7i}(x,z,t,w) = 0,$$

where d is the degree of X and f_i is quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil that is cut out on X by

$$\lambda x^8 + \mu z = 0,$$

where $(\lambda : \mu) \in \mathbb{P}^1$.

K3-Proposition 3.11.1. *A general surface in the pencil $\mathcal{P}$ is birational to a smooth K3 surface.*

Proof. If $\daleth = 84$, then a general surface in the pencil $\mathcal{P}$ is a compactification of a quartic surface in $\mathbb{C}^3$ and must be birational to a smooth K3 surface by Theorem 0.1.3. If $\daleth = 93$, then a general surface in the pencil $\mathcal{P}$ is a compactification of a double cover of $\mathbb{C}^2$ ramified along a sextic curve that must be birational to a smooth K3 surface. □

Proposition 3.11.2. *If $\daleth \in \{84, 93\}$, then the pencils $|-a_1K_X|$ and $\mathcal{P}$ are unique Halphen pencils on X.*

Proof. Theorem 0.2.4, Lemmas 0.3.3, 0.3.10, 0.3.11, and Corollary 0.3.7 immediately imply the result. □

3.12. Case $\gimel = 95$, hypersurface of degree 66 in $\mathbb{P}(1,5,6,22,33)$.

Let X be a general hypersurface of degree 66 in $\mathbb{P}(1,5,6,22,33)$ with $-K_X^3 = \frac{1}{330}$. Its singularities consist of a quotient singular point of type $\frac{1}{2}(1,1,1)$, a quotient singular point of type $\frac{1}{3}(1,2,1)$, a quotient singular point P of type $\frac{1}{5}(1,2,3)$, and a quotient singular point O of type $\frac{1}{11}(1,5,6)$.

We have an elliptic fibration as follows:

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,5,6) \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point O with weights $(1,5,6)$,
- η is an elliptic fibration.

The threefold X can be given by the equation

$$y^{12}z + \sum_{i=0}^{11} y^i f_{66-5i}\big(x,z,t,w\big) = 0$$

in $\mathbb{P}(1,5,6,22,33)$, where f_i is a quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil on the threefold X that is cut out by the pencil $\lambda x^6 = \mu z$, where $(\lambda : \mu) \in \mathbb{P}^1$. Then, Lemma 0.3.11 implies $\mathcal{P}$ is a Halphen pencil as well.

Proposition 3.12.1. *If $\gimel = 95$, then the linear systems $|-5K_X|$ and $\mathcal{P}$ are the only Halphen pencils.*

Proof. The proof is almost same as the cases $\gimel = 89$, 90, 92, 94. We have one thing different from these cases. It has a singular point at $(0:1:0:0:0)$ such that it gives us $b=6$, $c=0$, and $-K_Y^3 < 0$ for Proposition 0.3.9. The result therefore follows from Lemmas 2.13.1 and 0.3.11. □

K3-Proposition 3.12.2. *A general surface in the pencil $\mathcal{P}$ is birational to a K3 surface.*

Proof. Let $\alpha : Y \to X$ be the Kawamata blow up at the point P with weights $(1,2,3)$ and let E be its exceptional divisor. Then, the surface E is isomorphic to $\mathbb{P}(1,2,3)$. Let D be a general surface in the pencil $\mathcal{P}$. Then, the intersection $\Delta := E \cdot D_Y$ is a curve of degree six on E. It does not pass through any singular point of the surface E. We suppose that the curve Δ is smooth. Because it does not pass through any singular point of E and its degree on E is 6, it is an elliptic curve. The singularities of the surface D are rational except the point P. Then, the same argument of K3-Proposition 3.2.3 gives a contradiction. Therefore, the surface D_{V_1} is birational to a smooth K3 surface. □

Part 4. Fano threefold hypersurfaces with more than two Halphen pencils.

4.1. Case $\gimel = 18$, hypersurface of degree 12 in $\mathbb{P}(1,2,2,3,5)$.

The threefold X is a general hypersurface of degree 12 in $\mathbb{P}(1,2,2,3,5)$ with $-K_X^3 = \frac{1}{5}$. The singularities of X consist of six points O_1, O_2, O_3, O_4, O_5 and O_6 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and one point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram

$$\begin{tikzcd} U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\ X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,2), \end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point P with weights $(1,2,3)$,
- β is the Kawamata blow up with weights $(1,2,1)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1,2,1)$,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$w^2z + wf_7(x,y,z,t) + f_{12}(x,y,z,t) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces that are cut out on the hypersurface X by the equations $\lambda x^2 + \mu z = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$.

K3-Proposition 4.1.1. *A general surface of the pencil $\mathcal{P}$ is birational to a K3 surface. In particular, the linear system $\mathcal{P}$ is a Halphen pencil.*

Proof. A general surface of the pencil $\mathcal{P}$ is not ruled because X is birationally rigid ([7]). Hence, a general surface of the pencil $\mathcal{P}$ is birational to a K3 surface because it is a compactification of a double cover of $\mathbb{C}^2$ branched over a sextic curve. □

The hypersurface X can also be given by the equation

$$xg_{11}(x,y,z,t,w) + tg_9(x,y,z,t,w) + wg_7(x,y,z,t,w) + yg_5(y,z) = 0$$

such that the point O_1 is given by the equations $x = y = t = w = 0$, where g_i is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}_1$ be the pencil of surfaces that are cut out on the hypersurface X by the pencil $\lambda x^2 + \mu y = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. We will see that the linear system $\mathcal{P}_1$ is a Halphen pencil. The base locus of $\mathcal{P}_1$ does not contain the points O_2, O_3, O_4, O_5 and O_6. Similarly, we can construct a Halphen pencil $\mathcal{P}_i$ such that $\mathcal{P}_i \subset |-2K_X|$ and the base locus of the pencil $\mathcal{P}_i$ contains the point O_i.

Proposition 4.1.2. *The linear systems $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, and $\mathcal{P}_6$ are the only Halphen pencils on X.*

We may assume that the singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ are canonical. Moreover, it follows from Lemmas 0.3.3 and Corollary 0.3.7 that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{O_1, O_2, O_3, O_4, O_5, O_6, P\Big\}.$$

Lemma 4.1.3. *If $O_i \in \mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, then $\mathcal{M} = \mathcal{P}_i$.*

Proof. Let $\pi_i : V_i \to X$ be the Kawamata blow up at the point O_i with weights $(1,1,1)$. Then, $\mathcal{M}_{V_i} \sim_{\mathbb{Q}} -nK_{V_i}$ by Lemma 0.2.6.

The linear system $|-2K_{V_i}|$ is the proper transform of the pencil $\mathcal{P}_i$ and the base locus of $|-2K_{V_i}|$ consists of the irreducible curve C_{V_i} such that $\pi(C_{V_i})$ is the base curve of the pencil $\mathcal{P}_i$.

Let D be a general surface in $|-2K_{V_i}|$. Then, the surface D is normal and $C_{V_i}^2 < 0$ on the surface D. On the other hand, we have $C_{V_i} \equiv -K_{V_i}|_D$, which implies that $\mathcal{M}_{V_i} = |-2K_{V_i}|$ by Theorem 0.2.9. □

K3-Proposition 4.1.4. *A general surface of each pencil $\mathcal{P}_i$ is birational to a K3 surface. In particular, $\mathcal{P}_i$ is a Halphen pencil.*

Proof. We use the same notations as in the proof of Lemma 4.1.3. The pencil $|-2K_{V_i}|$ satisfies the condition of Theorem 0.2.10. Therefore, it is a Halphen pencil. The intersection of the surface D and the exceptional divisor $E_i \cong \mathbb{P}^2$ of the birational morphism π is a conic on E_i. An irreducible component of the intersection $D \cdot E_i$ is a rational curve not contained in the base locus of the pencil $|-2K_{V_i}|$. Therefore, the surface D is birational to a K3 surface by Corollary 0.2.12. □

Let E be the exceptional divisor of the birational morphism α. It has two singular points Q and O that are quotient singularities of types $\frac{1}{3}(1,2,1)$ and $\frac{1}{2}(1,1,1)$, respectively.

Let C be the base curve of the pencil $\mathcal{P}$ and L be the unique curve in of the linear system $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$ on E.

Lemma 4.1.5. *If the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ contains the point O, then $\mathcal{M} = \mathcal{P}$.*

Proof. Let $\pi : V \to U$ be the Kawamata blow up at the point O with weights $(1,1,1)$ and F be the exceptional divisor of the birational morphism π. Let $\mathcal{L}$ be the proper transform of the linear system $|-3K_U|$ by the birational morphism π. We have $\mathcal{M}_V \sim_{\mathbb{Q}} -nK_V$ by Lemma 0.2.6, $\mathcal{P}_V \sim_{\mathbb{Q}} -2K_V$, and

$$\mathcal{L} \sim_{\mathbb{Q}} \pi^*(-3K_U) - \frac{1}{2}F.$$

The base locus of the linear system $\mathcal{L}$ consists of the irreducible curve $\tilde{C}_V$. Moreover, for a general surface T of the linear system $\mathcal{L}$, the inequality $T \cdot \tilde{C}_V > 0$ holds, which implies that the divisor $\pi^*(-6K_U) - F$ is nef and big.

Let M and D be general surfaces of the pencils $\mathcal{M}_V$ and $\mathcal{P}_V$, respectively. Then,

$$\Big(\pi^*(-6K_U) - F\Big) \cdot M \cdot D = \Big(\pi^*(-6K_U) - F\Big) \cdot \Big(\pi^*(-nK_U) - \frac{n}{2}F\Big) \cdot \Big(\pi^*(-2K_U) - F\Big) = 0,$$

which implies that $\mathcal{M}_V = \mathcal{P}_V$ by Theorem 0.2.9. □

For now, to prove Proposition 4.1.2, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{P\}$. Because $\mathcal{M}_U \sim_{\mathbb{Q}} -nK_U$ by Lemma 0.2.6, the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M})_U$ is not empty by Theorem 0.2.4. Therefore, Lemma 4.1.5 enables us to assume that the set $\mathbb{CS}(U, \frac{1}{n}\mathcal{M}_U)$ consists of the point Q. The equivalence $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$ by Lemma 0.2.6 implies that every surface in the pencil $\mathcal{M}_W$ is contracted to a curve by the elliptic fibration η. Moreover, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ is not empty by Theorem 0.2.4.

Let G be the exceptional divisor of the birational morphism β and Q_1 be the singular point of the surface G. Then, the point Q_1 is the quotient singularity of type $\frac{1}{2}(1,1,1)$ on the variety W. Moreover, it follows from Lemma 0.2.7 that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q_1.

Let $\gamma : Y \to W$ be the Kawamata blow up at the point Q_1 with weights $(1,1,1)$. The base locus of the pencil $\mathcal{P}_Y$ consists of the irreducible curves C_Y and L_Y. Let D be a general surface of the pencil $\mathcal{P}_Y$. Then, explicit local calculations show that $D \sim_{\mathbb{Q}} -2K_Y$. On the other hand, the surface D is normal and the intersection form of the curves C_Y and L_Y on the surface D is negative-definite. Hence, we obtain the identity $\mathcal{M}_Y = \mathcal{P}_Y$ from Theorem 0.2.9 because $\mathcal{M}_Y|_D \equiv n(C_Y + L_Y)$. Therefore, we see that $\mathcal{M} = \mathcal{P}$, which completes our proof of Proposition 4.1.2.

4.2. Case $\gimel = 22$, hypersurface of degree 14 in $\mathbb{P}(1,2,2,3,7)$.

The threefold X is a general hypersurface of degree 14 in $\mathbb{P}(1,2,2,3,7)$ with $-K_X^3 = \frac{1}{6}$. The singularities of X consist of seven points O_1, O_2, O_3, O_4, O_5, O_6, and O_7 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and one point P that is a quotient singularity of type $\frac{1}{3}(1,2,1)$.

There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,2,2), \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point P with weights $(1,2,1)$,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$t^4z + t^3f_5(x,y,z,w) + t^2f_8(x,y,z,w) + tf_{11}(x,y,z,w) + f_{14}(x,y,z,w) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces that are cut out on the hypersurface X by the equations $\lambda x^2 + \mu z = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. We will see that the linear system $\mathcal{P}$ is a Halphen pencil.

The hypersurface X can also be given by the equation

$$xg_{13}(x,y,z,t,w) + tg_{11}(x,y,z,t,w) + wg_7(x,y,z,t,w) + yg_{12}(y,z) = 0$$

such that the point O_1 is given by the equations $x = y = t = w = 0$, where g_i is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}_1$ be the pencil of surfaces cut out on the hypersurface X by the pencil $\lambda x^2 + \mu y = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. Then, $\mathcal{P}_1$ is a Halphen pencil. Indeed, a general surface in this pencil is a compactification of a double cover of $\mathbb{C}^2$ branched over a sextic curve. The base locus of $\mathcal{P}_1$ does not contain the points O_2, O_3, O_4, O_5, O_6, and O_7. Similarly, we can construct a Halphen pencil $\mathcal{P}_i$ such that $\mathcal{P}_i \subset |-2K_X|$ and the base locus of the pencil $\mathcal{P}_i$ contains the point O_i.

Proposition 4.2.1. *The linear systems $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, and $\mathcal{P}_7$ are the only Halphen pencils on X.*

Proof. Due to Lemmas 0.3.3 and Corollary 0.3.7, we may assume that

$$\mathbb{CS}\Big(X, \frac{1}{n}\mathcal{M}\Big) \subset \Big\{O_1, O_2, O_3, O_4, O_5, O_6, O_7, P\Big\}.$$

If it contains the point O_i, we consider the Kawamata blow up $\pi_i : Y_i \to X$ at the point O_i with weights $(1,1,1)$. The proof of Lemma 4.1.3 then shows $\mathcal{M} = \mathcal{P}_i$.

From now, we suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of the point P. Then, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ must contain the singular point that is contained in the exceptional divisor of the birational morphism π.

Let $\sigma : W \to Y$ be the Kawamata blow up at this point. Then, the base locus of the pencil $\mathcal{P}_W$ consists of the irreducible curve $\bar{C}_W$. Moreover, $\mathcal{P}_W \sim_{\mathbb{Q}} -2K_W$, $-K_W \cdot \bar{C}_W < 0$, and $\mathcal{M}_W \sim_{\mathbb{Q}} -nK_W$. Therefore, Theorem 0.2.9 gives us the identity $\mathcal{M} = \mathcal{P}$. □

K3-Proposition 4.2.2. *A general surface in each of the pencils $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, and $\mathcal{P}_7$ is birational to a smooth K3 surface.*

Proof. A general surface in the pencil $\mathcal{P}_i$ is a compactification of a double cover of $\mathbb{C}^2$ ramified along a sextic curve. By Theorem 0.1.3, it must be birational to a smooth K3 surface. For the pencil $\mathcal{P}$, use the same argument with the exceptional divisor of σ as in K3-Proposition 4.1.4. □

4.3. Case $\gimel = 28$, hypersurface of degree 15 in $\mathbb{P}(1,3,3,4,5)$.

The threefold X is a general hypersurface of degree 15 in $\mathbb{P}(1,3,3,4,5)$ with $-K_X^3 = \frac{1}{12}$. The singularities of X consist of five points O_1, O_2, O_3, O_4, and O_5 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and one point P that is a quotient singularity of type $\frac{1}{4}(1,3,1)$.

There is a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,3,3), \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point P with weights $(1,3,1)$,
- η is an elliptic fibration.

The hypersurface X can be given by the equation

$$t^3z + t^2f_7(x,y,z,w) + tf_{11}(x,y,z,w) + f_{15}(x,y,z,w) = 0,$$

where $f_i(x,y,z,t)$ is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil of surfaces that are cut out on the hypersurface X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. As in the previous cases, the linear system $\mathcal{P}$ is a Halphen pencil.

The hypersurface X can also be given by the equation

$$xg_{14}(x,y,z,t) + tg_{11}(x,y,z,t) + wg_{10}(x,y,z,t) + yg_{12}(y,z) = 0$$

such that the point O_1 is given by the equations $x = y = t = w = 0$, where g_i is a general quasihomogeneous polynomial of degree i. Let $\mathcal{P}_1$ be the pencil of surfaces that are cut out on the hypersurface X by the pencil $\lambda x^3 + \mu y = 0$, where $(\lambda : \mu) \in \mathbb{P}^1$. Then, the linear system-$\mathcal{P}_1$ is a Halphen pencil. The base locus of $\mathcal{P}_1$ does not contain the points O_2, O_3, O_4, and O_5. Similarly, we can construct a Halphen pencil $\mathcal{P}_i$ such that $\mathcal{P}_i \subset |-3K_X|$ and the base locus of the pencil $\mathcal{P}_i$ contains the point O_i.

Proposition 4.3.1. *The linear systems $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$ are the only Halphen pencils on X.*

Proof. The proof is the same as that of Proposition 4.2.1. □

K3-Proposition 4.3.2. *A general surface in each of the pencils $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$ is birational to a smooth K3 surface.*

Proof. Essentially, the proof is the same as the proof of K3-Proposition 4.1.4. Instead of a conic, we however consider a cubic curve on $\mathbb{P}(1,1,2)$ or $\mathbb{P}(1,1,3)$, which has a rational irreducible component. □

Part 5. Fano threefold hypersurfaces with infinitely many Halphen pencils.

5.1. Case ℶ = 1, hypersurface of degree 4 in $\mathbb{P}^4$.

Let X be a general quartic hypersurface in $\mathbb{P}^4$. It is smooth and the log pair $(X, \frac{1}{n}\mathcal{M})$ is canonical (Theorem 3.6 in [6]).

Proposition 5.1.1. *Every Halphen pencil is contained in $|-K_X|$.*

Let us prove Proposition 5.1.1. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve Z and does not contain any point. Then, we have

$$\mathrm{mult}_Z(\mathcal{M}) = n.$$

It follows from Lemma 0.2.3 that $\deg(Z) \leq 4$.

Lemma 5.1.2. *The curve Z is contained in a two-dimensional linear subspace of $\mathbb{P}^4$.*

Proof. Suppose that the curve Z is not contained in any plane in $\mathbb{P}^4$. Then, the degree of the curve Z is either 3 or 4. If the degree is 3, then the curve is smooth. If the degree is 4, then the curve can be singular but the singularities consist of only one double point.

Suppose that Z is smooth. Let $\alpha : U \to X$ be the blow up along the curve Z and F be its exceptional divisor. Then, the base locus of the linear system $|\alpha^*(-\deg(Z)K_X) - F|$ does not contain any curve but

$$\Big(\alpha^*(-\deg(Z)K_X) - F\Big) \cdot D_1 \cdot D_2 < 0,$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}_U$, which is a contradiction.

Suppose that the curve Z is a quartic curve with a double point P. Let $\beta : W \to X$ be the composition of the blow up at the point P with the blow up along the proper transform of the curve Z. Let G and E be the exceptional divisors of β such that $\beta(E) = Z$ and $\beta(G) = P$. Then, the base locus of the linear system $|\beta^*(-4K_X) - E - 2G|$ does not contain any curve but

$$\Big(\beta^*(-4K_X) - E - 2G\Big) \cdot D_1 \cdot D_2 < 0$$

where D_1 and D_2 are general surfaces of the linear system $\mathcal{M}_W$, which is a contradiction. □

Lemma 5.1.3. *If the curve Z is a line, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.*

Proof. Let $\pi : V \to X$ be the blow up along the line Z. Then, the linear system $|-K_V|$ is base-point-free and induces an elliptic fibration $\eta : V \to \mathbb{P}^2$. Therefore, $\mathcal{M}_V$ is contained in fibers of η. In particular, the base locus of the pencil $\mathcal{M}_V$ does not contain curves not contracted by the morphism η.

The set $\mathbb{CS}(V, \frac{1}{n}\mathcal{M}_V)$ is not empty by the Theorem 0.2.4. However, it does not contain any point because we assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain points. Hence, there is an irreducible curve $L \subset V$ such that $\mathrm{mult}_L(\mathcal{M}_V) = n$ and $\eta(L)$ is a point.

The pencil $\mathcal{M}_V$ is the pull-back of a pencil $\mathcal{P}$ on $\mathbb{P}^2$ via the morphism η such that $\mathcal{P} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^2}(n)$. Hence, the equality $\mathrm{mult}_L(\mathcal{M}_V) = n$ implies that the multiplicity of the pencil $\mathcal{P}$ at the point $\eta(L)$ is n, which implies that $n = 1$. □

Thus, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain lines. Moreover, the pencil $\mathcal{M}$ is contained in $|-K_X|$ if Z is a plane quartic curve by Theorem 0.2.9. Thus, we may assume that Z is either a plane cubic curve or a conic.

Lemma 5.1.4. *If the curve Z is a cubic, then $\mathcal{M}$ is a pencil in $|-K_X|$.*

Proof. Let $\mathcal{P}$ be the pencil in $|-K_X|$ that contains all surfaces passing through the cubic curve Z and D be a general surface in $\mathcal{P}$. Then, D is a smooth K3 surface but the base locus of the pencil $\mathcal{P}$ consists of the curve Z and some line $L \subset X$. We have

$$\mathcal{M}\Big|_D = nZ + \mathrm{mult}_L(\mathcal{M})L + \mathcal{B} \equiv nZ + nL,$$

where $\mathcal{B}$ is a pencil on D without fixed components. On the other hand, we have $L^2 = -2$ on the surface D, which implies that $\mathrm{mult}_L(\mathcal{M}) = n$ and $\mathcal{B} = \varnothing$. Hence, we have $\mathcal{M} = \mathcal{P}$ by Theorem 0.2.9. □

Therefore, we may assume that the curve Z is a conic. Let Π be the plane in $\mathbb{P}^4$ that contains the conic Z.

Lemma 5.1.5. *If $\Pi \cap X = Z$, then $\mathcal{M}$ is a pencil in $|-K_X|$.*

Proof. Let $\alpha : U \to X$ be the blow up along the curve Z and D be a general surface of the pencil $|-K_U|$. Then, D is a smooth K3 surface but the base locus of the pencil $|-K_U|$ consists of the irreducible curve L such that $\alpha(L) = Z$ and $-K_U|_D \equiv L$. Therefore, we have

$$\mathcal{M}_U\Big|_D \equiv nL,$$

but $L^2 = -2$ on the surface D. Hence, we have $\mathcal{M}_U = |-K_U|$ by Theorem 0.2.9. □

In the case when the set-theoretic intersection $\Pi \cap X$ contains a curve different from a conic Z, the arguments of the proof of Lemma 5.1.4 easily imply that $\mathcal{M}$ is a pencil in $|-K_X|$. Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a point P of the quartic X.

Let M_1 and M_2 be two general surfaces in $\mathcal{M}$. Then, the inequality $\mathrm{mult}_P(M_1 \cdot M_2) \geq 4n^2$ holds ([6] and [17]). On the other hand, the degree of the cycle $M_1 \cdot M_2$ is $4n^2$, which implies that $\mathrm{mult}_P(M_1 \cdot M_2) = 4n^2$. In particular, the support of the cycle $M_1 \cdot M_2$ consists of the union of all lines passing through the point P, which implies that there are at most finitely many lines on the quartic X passing through the point P. Moreover, the equality $\mathrm{mult}_P(\mathcal{M}) = 2n$ holds ([1], [2], Corollary C.14 in [3]).

Lemma 5.1.6. *For a line L on X passing through P, $\frac{n}{3} \leq \mathrm{mult}_L(\mathcal{M}) \leq \frac{n}{2}$.*

Proof. Let D be a general hyperplane section of X that passes through the line L and M be a general surface in $\mathcal{M}$. Then, D is a smooth K3 surface and

$$M\Big|_D = \mathrm{mult}_L(\mathcal{M})L + \Delta,$$

where Δ is an effective divisor such that $\mathrm{mult}_P(\Delta) \geq 2n - \mathrm{mult}_L(\mathcal{M})$. On the other hand, we have $L^2 = -2$ on the surface D. Hence, we have

$$n + 2\mathrm{mult}_L(\mathcal{M}) = L \cdot \Delta \geq \mathrm{mult}_P(\Delta) \geq 2n - \mathrm{mult}_L(\mathcal{M}),$$

which implies $\frac{n}{3} \leq \mathrm{mult}_L(\mathcal{M})$.

Let T be the hyperplane section tangent to the quartic X at the point P. Then, T has isolated singularities and the point P is an isolated double point of the surface T because $\mathrm{mult}_P(M) = 2n$. The cycle $T \cdot D$ is reduced and consists of the line L and possibly reducible cubic curve $Z \subset D$ that passes through the point P. Thus, we have

$$3n = \big(\mathrm{mult}_L(\mathcal{M})L + \Delta\big) \cdot Z = 3\mathrm{mult}_L(\mathcal{M}) + \Delta \cdot Z \geq 3\mathrm{mult}_L(\mathcal{M}) + 2n - \mathrm{mult}_L(\mathcal{M}),$$

which implies $\mathrm{mult}_L(\mathcal{M}) \leq \frac{n}{2}$. □

Therefore, any curve containing the point P cannot belong to the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$. Let $\pi : V \to X$ be the blow up at the point P and E be the exceptional divisor of the blow up π. In addition, let B_i be the proper transform of the divisor M_i by π. Then, the equalities $\mathrm{mult}_P(M_1 \cdot M_2) = 4n^2$ and $\mathrm{mult}_P(\mathcal{M}) = 2n$ imply that

$$B_1 \cdot B_2 = \sum_{i=1}^{k} \mathrm{mult}_{\bar{L}_i}\big(B_1 \cdot B_2\big)\bar{L}_i,$$

where k is a number of lines on X that passes through the point P and $\bar{L}_i$ is an irreducible curve such that $\pi(\bar{L}_i)$ is a line on X that passes through the point P.

Lemma 5.1.7. *Let Z be an irreducible curve on X that is not a line passing through the point P. Then,*

$$\deg(Z) \geq 2\mathrm{mult}_P(Z),$$

where the equality holds only if the proper transform Z_V does not intersect the curve $\bar{L}_i$ for any i.

Proof. The proper transform Z_V is not contained in B_i because the base locus of the pencil $\mathcal{M}_V$ consists of the curves $\bar{L}_1, \cdots, \bar{L}_k$. Hence, we have

$$0 \leq B_i \cdot Z_V \leq n\big(\deg(Z) - 2\mathrm{mult}_P(Z)\big),$$

which concludes the proof. □

Note that so far we never use the generality of the quartic X besides its smoothness. In the following we assume that there are at most 3 lines on X passing though a given point of X and every line on X has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. It follows from the proof of Proposition 1 in [17] that the former condition is satisfied on a general quartic threefold. The latter condition is also satisfied on a general quartic threefold by [5]. Moreover, the article [5] shows that the latter condition is equivalent to the following: no two-dimensional linear subspace of $\mathbb{P}^4$ is tangent to the quartic X along a line. In particular, we see that no hyperplane section of X can be singular at three points that are contained in a single line.

Lemma 5.1.8. *For a line L in X passing through P,* $\mathrm{mult}_L(\mathcal{M}) = \frac{n}{2}$.

Proof. By Lemma 5.1.6, it is enough to show $\mathrm{mult}_L(\mathcal{M}) \geq \frac{n}{2}$.

Let $\alpha : W \to X$ be the blow up along the line L and F be the exceptional divisor of the blow up α. Then, the surface F is the rational ruled surface $\mathbb{F}_1$.

Let Δ be the irreducible curve on the surface F such that $\Delta^2 = -1$ and Z be the fiber of the restricted morphism $\pi|_F : F \to L$ over the point P. Then, $F|_F \equiv -(\Delta + Z)$, which implies that

$$\mathcal{M}_W\Big|_F \equiv nZ + \mathrm{mult}_L(\mathcal{M})(\Delta + Z).$$

Let $\beta : U \to W$ be the blow up along the curve Z and G be the exceptional divisor of β. Then, the exceptional divisor E of π is the proper transform of the divisor G on the threefold V. Hence, we have

$$\mathrm{mult}_Z(\mathcal{M}_W) = 2n - \mathrm{mult}_L(\mathcal{M}),$$

which implies that $\mathrm{mult}_Z(\mathcal{M}_W|_F) \geq 2n - \mathrm{mult}_L(\mathcal{M})$. Therefore, we have

$$n + \mathrm{mult}_L(\mathcal{M}) \geq 2n - \mathrm{mult}_L(\mathcal{M}),$$

which gives $\mathrm{mult}_L(\mathcal{M}) \geq \frac{n}{2}$. □

Let T be a hyperplane section of X that is singular at the point P. Then, T has only isolated singularities. Moreover, we have $\mathrm{mult}_P(T \cdot M_i) = 4n$, which implies that the point P is an isolated double point of the surface T. Put $L_i = \pi(\bar{L}_i)$.

Lemma 5.1.9. *The point P is not an ordinary double point of the surface T.*

Proof. Suppose that the point P is an ordinary double point of the surface T. Let us show that this assumption leads us to a contradiction.

Let H_i be a general hyperplane section of the quartic X that passes through the line L_i. Then,

$$H_i \cdot T = L_i + Z_i,$$

where Z_i is a cubic curve. The cubic curve Z_i intersect the line L_i at the point P and at some smooth point of the surface T because L_i does not contain three singular points of the surface T. Hence, we have

$$L_i^2 = H_i \cdot L_i - Z_i \cdot L_i < -\frac{1}{2}.$$

The proper transform T_V has isolated singularities and normal. Moreover, the inequality $L_i^2 < -\frac{1}{2}$ implies that $\bar{L}_i^2 < -1$.

Let M be a general surface in $\mathcal{M}$. The support of the cycle $T \cdot M$ consists of the union of all lines on X passing through the point P because $\mathrm{mult}_P(T \cdot M) = 4n$. Thus, the equalities $\mathrm{mult}_P(T) = 2n$ and $\mathrm{mult}_P(M) = 2n$ implies that the support of the cycle $T_V \cdot M_V$ consists of the union of the curves $\bar{L}_1, \cdots, \bar{L}_k$. Hence, we have

$$M_V\Big|_{T_V} = \sum_{i=1}^{k} m_i \bar{L}_i,$$

but $M_V \cdot \bar{L}_l = -n$ and $\bar{L}_i \cdot \bar{L}_j = 0$ for $i \neq j$. Hence, we have

$$-n = M_V \cdot \bar{L}_j = \sum_{i=1}^{k} m_i \bar{L}_i \cdot \bar{L}_j = m_j \bar{L}_j^2,$$

which implies that $m_j < n$.

Let H be the proper transform of a general hyperplane section of X on the threefold V. Then,

$$4n = M_V \cdot T_V \cdot H = \sum_{i=1}^{k} m_i \bar{L}_i \cdot H = \sum_{i=1}^{k} m_i < kn,$$

which implies that $k > 4$. Thus, the threefold X has at least five lines that pass through the point P, which is a contradiction. $\square$

Thus, the point P is not an ordinary double point on the surface T. Therefore, there is a hyperplane section Z of the quartic surface T with $\mathrm{mult}_P(Z) \geq 3$. Hence, the curve Z is reducible by Lemma 5.1.7. Moreover, the curve Z is reduced and $\mathrm{mult}_P(Z) = 3$ by our assumption of generality of the quartic X.

Lemma 5.1.10. *The curve Z is not a union of four lines.*

Proof. Suppose that the curve Z is a union of four lines. Then, one component of Z is a line L that does not pass through the point P. Then, L intersects M_i at least three points that are contained in the union of the lines $L_1, \cdots, L_k$. On the other hand, we have $M_i \cdot L = n$, which implies that L is contained in M_i by Lemma 5.1.8, which is impossible because the base locus of $\mathcal{M}$ is the union of the lines $L_1, \cdots, L_k$. $\square$

The curve Z is not a union of an irreducible cubic curve and a line due to Lemma 5.1.7. Hence, the curve Z is a union of two different lines passing through the point P and a conic that also passes through the point P, which is impossible by Lemma 5.1.7. Hence, we have completed the proof of Proposition 5.1.1.

5.2. Case $\daleth = 2$, hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$.

The threefold X is a general hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$ with $-K_X^3 = \frac{5}{2}$. It has only one singular point O at $(0:0:0:0:1)$ which is a quotient singularity of type $\frac{1}{2}(1,1,1)$. The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z,t) + w f_3(x,y,z,t) + f_5(x,y,z,t) = 0,$$

where f_i is a homogeneous polynomial of degree i.

There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & W_i \arrow[ll, "\alpha_i"'] \arrow[dr, "w_i"] & \\
X \arrow[dr, dashed, "\psi"'] & & Y' \arrow[dl, "\omega"] & & U_i \arrow[ll, "\beta_i"'] \arrow[dl, "\eta_i"] \\
 & \mathbb{P}^3 \arrow[rr, dashed, "\chi_i"'] & & \mathbb{P}^2 &
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point O with weights $(1, 1, 1)$,
- γ is the birational morphism that contracts 15 smooth rational curves $L_1, \cdots, L_{15}$ to 15 isolated ordinary double points $P_1, \cdots, P_{15}$ of the variety Y', respectively,
- α_i is the blow up along the curve L_i,
- β_i is the blow up at the point P_i,
- w_i is a birational morphism,
- ω is a double cover of $\mathbb{P}^3$ branched over a sextic surface $R \subset \mathbb{P}^3$,
- χ_i is the projection from the point $\omega(P_i)$,
- η_i is an elliptic fibration.

The surface R is given by the equation

$$f_3\big(x, y, z, t\big)^2 - 4f_1\big(x, y, z, t\big)f_5\big(x, y, z, t\big) = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t]\Big).$$

It has 15 ordinary double points $\omega(P_1), \cdots, \omega(P_{15})$ that are given by the equations

$$f_3\big(x, y, z, t\big) = f_1\big(x, y, z, t\big) = f_5\big(x, y, z, t\big) = 0 \subset \mathbb{P}^3.$$

We may assume that the curves on $\mathbb{P}^3$ defined by $f_3 = f_1 = 0$ and $f_5 = f_1 = 0$ are irreducible.

For the convenience, let M and M' be general surfaces in the pencil $\mathcal{M}$.

Lemma 5.2.1. *The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ does not contain any smooth point of X.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a smooth point P of X. Let D be a general surface of the linear system $|-K_X|$ that passes through the point P. The surface D does not contain an irreducible component of the cycle $M \cdot M'$ if none of $\pi(L_i)$ passes through the point P. In particular, in such a case, we see

$$\mathrm{mult}_P\Big(M \cdot M'\Big) \leq M \cdot M' \cdot D = -n^2K_X^3 = \frac{5}{2}n^2,$$

which is impossible by [17]. Thus, we may assume that the curve $\pi(L_1)$ passes through the point P.

Let us use the arguments of the article [7]. Put $L = \pi(L_1)$ and

$$\mathcal{M}\Big|_D = \mathcal{L} + \mathrm{mult}_L\big(\mathcal{M}\big)L,$$

where $\mathcal{L}$ is a pencil on the surface D without fixed curves. Then, the point P is a center of log canonical singularities of the log pair $(D, \frac{1}{n}\mathcal{M}|_D)$ by the Shokurov connectedness principle ([6] and [21]). It implies that

$$\mathrm{mult}_P\Big(\Lambda_1 \cdot \Lambda_2\Big) \geq 4n\Big(n - \mathrm{mult}_L\big(\mathcal{M}\big)\Big)$$

by Theorem 3.1 in [6], where Λ_1 and Λ_2 are general curves in $\mathcal{L}$. The equality

$$\Lambda_1 \cdot \Lambda_2 = \frac{5}{2}n^2 - \mathrm{mult}_L\big(\mathcal{M}\big)n - \frac{3}{2}\mathrm{mult}_L^2\big(\mathcal{M}\big)$$

holds on the surface D because $L^2 = -\frac{3}{2}$ on the surface D. Hence, we have

$$\frac{5}{2}n^2 - \mathrm{mult}_L\big(\mathcal{M}\big)n - \frac{3}{2}\mathrm{mult}_L^2\big(\mathcal{M}\big) \geq 4n\Big(n - \mathrm{mult}_L\big(\mathcal{M}\big)\Big),$$

which gives $\mathrm{mult}_L(\mathcal{M}) = n$. Thus, the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the curve $\pi(L_1)$.

The set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point O by Lemma 0.2.6. Then, $\mathcal{M}_{W_1} \sim_{\mathbb{Q}} -nK_{W_1}$ because $\mathrm{mult}_L(\mathcal{M}) = n$, which implies that each surface of $\mathcal{M}_{W_1}$ is contracted to a curve by the elliptic fibration $\eta_1 \circ w_1$. On the other hand, the set $\mathbb{CS}(W_1, \frac{1}{n}\mathcal{M}_{W_1})$ contains a subvariety of the threefold W_1 that dominates the point P.

Let E_1 be the exceptional divisor of α_1. Then, $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the pencil $\mathcal{M}_{W_1}|_{E_1}$ does not have fixed components because E_1 is a section of the elliptic fibration $\eta_1 \circ \omega_1$ and the base locus of the pencil $\mathcal{M}_{W_1}$ does not contain curves not contracted by the elliptic fibration $\eta_1 \circ w_1$. Thus, the set $\mathbb{CS}(W_1, \frac{1}{n}\mathcal{M}_{W_1})$ contains a point Q of the surface E_1 such that $\pi \circ \alpha_1(Q) = P$.

The point Q is a center of log canonical singularities of the log pair $(E_1, \frac{1}{n}\mathcal{M}_{W_1}|_{E_1})$ by the Shokurov connectedness principle ([6] and [21]). Let Δ_1 and Δ_2 be general curves in $\mathcal{M}_{W_1}|_{E_1}$. Then, the inequality

$$2n^2 = \mathrm{mult}_Q\Big(\Delta_1 \cdot \Delta_2\Big) \geq 4n^2$$

holds by Theorem 3.1 in [6], which is a contradiction. □

Lemma 5.2.2. *If the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains a curve Λ not passing through the singular point O, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.*

Proof. We have $\mathrm{mult}_\Lambda(\mathcal{M}) = n$ and $-K_X \cdot \Lambda \leq 2$ by Corollary 0.3.4.

Suppose that $-K_X \cdot \Lambda = 2$ and $\psi(\Lambda)$ is a line. Then, the line $\psi(\Lambda)$ passes through a unique singular point of R. Hence, we may assume that the curve Λ intersects $\pi(L_i)$ only for $i = 1$.

Let $\mathcal{D}$ be the pencil in the linear system $|-K_X|$ consisting of surfaces that pass through the curve Λ and D be a general surface of the pencil $\mathcal{D}$. Then, the surface D is smooth in the outside of the singular point O, the point O is an ordinary double point of the surface D, and the base locus of the pencil $\mathcal{D}$ consists of the curve Λ and the curve $\pi(L_1)$. Put $L = \pi(L_1)$. Then,

$$\mathcal{M}\Big|_D = \mathrm{mult}_\Lambda\big(\mathcal{M}\big)\Lambda + \mathrm{mult}_L\big(\mathcal{M}\big)L + \mathcal{L} \equiv n\big(\Lambda + L\big),$$

where $\mathcal{L}$ is a pencil with no fixed curves. It gives $\mathcal{M} = \mathcal{D}$ by Theorem 0.2.9 because the inequality $L^2 < 0$ holds on the surface D.

We may assume that either the equality $-K_X \cdot \Lambda = 1$ holds or $\psi(\Lambda)$ is a conic, which implies that Λ is smooth. Let $\sigma : \breve{X} \to X$ be the blow up along the curve Λ and G be its exceptional divisor.

Suppose that $-K_X \cdot \Lambda = 2$. Then, Λ is cut, in the set-theoretic sense, by the surfaces of the linear system $|-2K_X|$ that pass through the curve Λ. Moreover, the scheme-theoretic intersection of two general surfaces of the linear system $|-2K_X|$ passing through the curve Λ is reduced at a generic point of the curve Λ, which implies that the divisor $\sigma^*(-2K_X) - G$ is nef by Lemma 5.2.5 in [7]. However, we obtain an absurd inequality

$$-3n^2 = \Big(\sigma^*(-2K_X) - G\Big) \cdot M_{\breve{X}} \cdot M'_{\breve{X}} \geq 0.$$

Therefore, the equality $-K_X \cdot \Lambda = 1$ holds, which implies that $|-K_{\breve{X}}|$ is a pencil.

Suppose that $\psi(\Lambda)$ is not contained in the plane $f_1(x, y, z, t) = 0$. Then, $\psi(\Lambda)$ contains a unique singular point of the surface $R \subset \mathbb{P}^3$. Hence, we may assume that the curve Λ intersects $\pi(L_i)$ only for $i = 1$. It implies that the base locus of the linear system $|-K_{\breve{X}}|$ consists of the irreducible curves $\breve{\Lambda}$ and $\breve{L}_1$ such that $(\psi \circ \sigma)(\breve{\Lambda}) = \psi(\Lambda)$ and $\sigma(\breve{L}_1) = \pi(L_1)$. Let $\breve{D}$ be a general surface in $|-K_{\breve{X}}|$. Then, we can consider the curves $\breve{\Lambda}$ and $\breve{L}_1$ as divisors on $\breve{D}$. We have

$$\breve{\Lambda}^2 = -2, \ \breve{L}_1^2 = -\frac{3}{2}, \ \breve{\Lambda} \cdot \breve{L}_1 = 1,$$

which implies the negative-definiteness of the intersection form of $\breve{\Lambda}$ and $\breve{L}_1$. Because

$$\mathcal{M}_{\breve{X}}\Big|_{\breve{D}} \equiv -nK_{\breve{X}}\Big|_{\breve{D}} \equiv n\big(\breve{\Lambda}+\breve{L}_1\big),$$

it follows from Theorem 0.2.9 that $\mathcal{M}_{\breve{X}} = |-K_{\breve{X}}|$.

Finally, we suppose that the line $\psi(\Lambda)$ is contained in the plane $f_1(x,y,z,t)=0$. In particular, the line $\psi(\Lambda)$ is not contained in the surface R because the curve $f_3=f_1=0$ is irreducible. Moreover, the line $\psi(\Lambda)$ contains exactly three singular points of the ramification surface[7]; otherwise the point O would belong to the curve Λ. Thus, the curve Λ intersects exactly three curves among the curves $L_1,\cdots,L_{15}$; otherwise Λ would contain the point O.

We may assume that Λ intersects the curves $\pi(L_1)$, $\pi(L_2)$, and $\pi(L_3)$, which means that the points $\omega(P_1),\omega(P_2),\omega(P_3)$ are contained in $\psi(\Lambda)$. The base locus of $|-K_{\breve{X}}|$ consists of the curves $\breve{L}_1$, $\breve{L}_2$, $\breve{L}_3$ such that $\sigma(\breve{L}_i)=\pi(L_i)$. The curves $\breve{L}_1$, $\breve{L}_2$, $\breve{L}_3$ can be contracted on the surface $\breve{D}$ to a singular point of type $\mathbb{D}_4$, which implies that their intersection form is negative-definite. Hence, we have $\mathcal{M}_{\breve{X}} = |-K_{\breve{X}}|$ by Theorem 0.2.9 . □

The equivalence $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$ holds by Lemma 0.2.6. It implies that the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains no point of Y due to Lemmas 0.2.7 and 5.2.1. Let $\mathcal{M}_{Y'}$ be the push-forward of the pencil $\mathcal{M}_Y$ by the birational morphism γ. Then, $\mathcal{M}_{Y'} \sim_{\mathbb{Q}} -nK_{Y'}$, the log pair $(Y',\frac{1}{n}\mathcal{M}_{Y'})$ has canonical singularities but it follows from Theorem 0.2.4 that the singularities of the log pair $(Y',\frac{1}{n}\mathcal{M}_{Y'})$ are not terminal.

Lemma 5.2.3. *If the set $\mathbb{CS}(Y',\frac{1}{n}\mathcal{M}_{Y'})$ contains an irreducible curve Γ with $-K_{Y'}\cdot\Gamma\neq 1$, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.*

Proof. Let D be a general divisor in $|-K_{Y'}|$. In addition, let $M_{Y'}=\gamma(M_Y)$ and $M'_{Y'}=\gamma(M'_Y)$. Then,

$$2n^2 = D\cdot M_{Y'}\cdot M'_{Y'} \geq \mathrm{mult}_{\Gamma}\Big(M_{Y'}\cdot M'_{Y'}\Big)D\cdot\Gamma \geq -n^2K_{Y'}\cdot\Gamma$$

because $\mathrm{mult}_{\Gamma}(\mathcal{M}_{Y'})=n$. Therefore, the inequality $-K_{Y'}\cdot\Gamma\leq 2$ holds.

Suppose that $-K_{Y'}\cdot\Gamma=2$ but the curve $\omega(\Gamma)$ is a line. Let $\mathcal{T}$ be the linear subsystem of the linear system $|-K_{Y'}|$ consisting of surfaces passing through the curve Γ and T be a general surface in the pencil $\mathcal{T}$. Then, the base locus of the pencil $\mathcal{T}$ consists of the curve Γ and the rational map induced by the pencil $\mathcal{T}$ is the composition of the double cover ω with the projection from the line $\omega(\Gamma)$. On the other hand, we have

$$2n = D\cdot T\cdot M_{Y'} \geq \mathrm{mult}_{\Gamma}\Big(T\cdot M_{Y'}\Big)D\cdot\Gamma \geq -nK_{Y'}\cdot\Gamma,$$

which implies that the support of the cycle $T\cdot M_{Y'}$ is contained in Γ. Thus, we have $\mathcal{M}_{Y'}=\mathcal{T}$ by Theorem 0.2.9.

For now, we suppose that $-K_{Y'}\cdot\Gamma=2$ but the curve $\omega(\Gamma)$ is a conic. Then, Γ is smooth and $\omega|_{\Gamma}$ is an isomorphism. Moreover, the curve Γ contains at most 2 singular points of the threefold Y' if the curve $\omega(\Gamma)$ is not contained in the plane $f_1(x,y,z,t)=0$, and the curve Γ contains at most 6 singular points of the threefold Y' otherwise. We may assume that Γ passes through $P_1,\cdots,P_k$, where $0\leq k\leq 6$. The equality $k=0$ means that Γ lies in the smooth locus of the threefold Y'.

Let $\beta: V\to Y'$ be the blow up at the points $P_1,\cdots,P_k$, and E_i be the exceptional divisor of the blow up β with $\beta(E_i)=P_i$. The exceptional divisor E_i is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. The proper transform Γ_V intersects the surface E_i transversally at a single point, which we denote by Q_i.

Let $\upsilon: W\to V$ be the blow up along the curve Γ_V and G be the exceptional divisor of the birational morphism υ. In addition, let A_i and B_i be the fibers of the natural projections of the surface E_i that pass through the point Q_i, and $\bar{A}_i$ and $\bar{B}_i$ be the proper transforms of the curves A_i and B_i on the threefold W, respectively. Then, we can flop the curves $\bar{A}_i$ and $\bar{B}_i$.

[7] In fact, we may assume that no three points of the set $\mathrm{Sing}(R)$ are collinear.

Let $\upsilon_1 : U \to W$ be the blow up along the curves $\bar{A}_1, \bar{B}_1, \cdots, \bar{A}_k, \bar{B}_k$. Also, let F_i and H_i be the exceptional divisors of υ_1 such that $\upsilon_1(F_i) = \bar{A}_i$ and $\upsilon_1(H_i) = \bar{B}_i$. Then, all the exceptional divisors are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There is a birational morphism $\upsilon_1' : U \to W'$ such that $\upsilon_1'(F_i)$ and $\upsilon_1'(H_i)$ are rational curves but $\upsilon_1' \circ \upsilon_1^{-1}$ is not biregular in a neighborhood of $\bar{A}_i$ and $\bar{B}_i$. Let E_i' be the proper transform of E_i on the threefold W'. Then, we can contract the surface E_i' to a singular point of type $\frac{1}{2}(1,1,1)$.

Let $\upsilon' : W' \to V'$ be the contraction of $E_1', \cdots, E_k'$ and G' be the proper transform of the surface G on the threefold V'. Then, there is a birational morphism $\beta' : V' \to Y'$ that contracts the divisor G' to the curve Γ. Hence, we constructed the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "\upsilon"'] & & U \arrow[ll, "\upsilon_1"'] \arrow[dr, "\upsilon_1'"] & \\ V \arrow[dr, "\beta"'] & & & & W' \arrow[dl, "\upsilon'"] \\ & Y' & & V'. \arrow[ll, "\beta'"'] & \end{tikzcd}$$

The threefold V' is projective. Its singularities consist of $15-k$ ordinary double points and k singular points of type $\frac{1}{2}(1,1,1)$. However, it is not $\mathbb{Q}$-factorial because the threefold Y' is not $\mathbb{Q}$-factorial.

The construction of the birational morphism β' implies that

$$\mathcal{M}_{V'} \sim_{\mathbb{Q}} -n\beta'^{*}(K_{Y'}) - nG' \sim_{\mathbb{Q}} -nK_{V'}.$$

Let D' be a general surface of the linear system $|\beta'^{*}(-4K_{Y'}) - G'|$. Then, the divisor D' is nef by Lemma 5.2.5 in [7]. The construction of the birational morphism β' implies that

$$0 > \Big(-4 + \frac{k}{2}\Big)n^2 = \Big(\beta'^{*}(-4K_{Y'}) - G'\Big) \cdot \Big(\beta'^{*}(-nK_{Y'}) - nG'\Big)^2 = D' \cdot M_{V'} \cdot M_{V'}' \geq 0,$$

where $M_{V'}$ and $M_{V'}'$ are the proper transforms of $M_{Y'}$ and $M_{Y'}'$ by the birational morphism β'. We have obtained a contradiction. $\square$

Lemma 5.2.4. *If the set $\mathbb{CS}(Y', \frac{1}{n}\mathcal{M}_{Y'})$ contains a curve Γ with $-K_Z \cdot \Gamma = 1$, then the pencil $\mathcal{M}$ is contained in $|-K_X|$.*

Proof. The curve $\omega(\Gamma)$ is a line in $\mathbb{P}^3$. The restricted morphism $\omega|_{\Gamma} : \Gamma \to \omega(\Gamma)$ is an isomorphism. The curve Γ contains at most one singular point of Y' if $\omega(\Gamma)$ is not contained in the plane $f_1(x,y,z,t) = 0$, and the curve $\omega(\Gamma)$ contains at most three singular points of the threefold Y' otherwise. We may assume that Γ contains $P_1, \cdots, P_k$, where $0 \leq k \leq 3$. Here, the equality $k = 0$ means that Γ lies in the smooth locus of the threefold Y'.

Suppose that the line $\omega(\Gamma)$ is not contained in R. Let D be a general surface in $|-K_{Y'}|$ that passes through the curve Γ. Then,

$$\mathcal{M}_{Y'}\Big|_D = \mathrm{mult}_{\Gamma}\big(\mathcal{M}_{Y'}\big)\Gamma + \mathrm{mult}_{\Omega}\big(\mathcal{M}_{Y'}\big)\Omega + \mathcal{L},$$

where $\mathcal{L}$ is a pencil without fixed curves and Ω is a smooth rational curve different from Γ such that $\omega(\Omega) = \omega(\Gamma)$. Moreover, the surface D is smooth in the outside the points $P_1, \cdots, P_k$ but the points $P_1, \cdots, P_k$ are isolated ordinary double points of the surface D. We have $\Omega^2 = -2 + \frac{k}{2}$ on the surface D and

$$\Big(n - \mathrm{mult}_{\Omega}(\mathcal{M}_{Y'})\Big)\Omega^2 = \Big(\mathrm{mult}_{\Gamma}(\mathcal{M}_{Y'}) - n\Big)\Gamma \cdot \Omega + L \cdot \Omega = L \cdot \Omega \geq 0,$$

where L is a general curve in $\mathcal{L}$. Therefore, the equality $\mathrm{mult}_{\Omega}(\mathcal{M}_{Y'}) = n$ holds, which easily implies that $\mathcal{M}_{Y'}$ is a pencil in $|-K_{Y'}|$ due to Theorem 0.2.9.

Finally, we suppose that $\omega(\Gamma)$ is contained in the ramification surface of ω. It implies that $\omega(\Gamma)$ is not contained in the plane $f_1(x,y,z,y) = 0$. The proof of Lemma 5.2.3 shows the existence of a birational morphism $\upsilon' : W' \to V'$ that contracts a single irreducible divisor G' to

the curve Γ, the surface G' contains k singular points of the threefold V' of type $\frac{1}{2}(1,1,1)$, and v' is the blow up of Γ at a generic point of Γ.

Let D' be a general surface in $|-K_{V'}|$. Then, $\omega\circ\beta'(D')$ is a plane that passes through $\omega(\Gamma)$. It implies that the base locus of the pencil $|-K_{V'}|$ consists of an irreducible curve Γ' such that $\beta'(\Gamma')=\Gamma$ and

$$D'\cdot\Gamma' = -K_{V'}^3 = -2+\frac{k}{2}.$$

Then, one can easily see that $\mathcal{M}_{V'}=|-K_{V'}|$ by Theorem 0.2.9. Hence, the linear system $\mathcal{M}$ is a pencil in $|-K_X|$. □

Proposition 5.2.5. *Every Halphen pencil is contained in $|-K_X|$.*

Proof. Let $\mathcal{M}_{U_i}$ be the push-forward of the pencil $\mathcal{M}_{W_i}$ by the morphism w_i. Due to the previous arguments, we may assume that

$$P_1\in\mathbb{CS}\Big(Y',\frac{1}{n}\mathcal{M}_{Y'}\Big)\subseteq\Big\{P_1,\cdots,P_{15}\Big\},$$

which implies that $\mathcal{M}_{U_1}\sim_{\mathbb{Q}}-nK_{U_1}$ by Theorem 3.10 in [6]. Therefore, each member in the pencil $\mathcal{M}_{U_1}$ is contracted to a curve by the elliptic fibration η_1. Therefore, the base locus of the pencil $\mathcal{M}_{U_1}$ does not contain curves that are not contracted by η_1. On the other hand, the singularities of the log pair $(U_1,\frac{1}{n}\mathcal{M}_{U_1})$ are not terminal by Theorem 0.2.4.

The proof of Lemma 5.2.1 implies that the set $\mathbb{CS}(U_1,\frac{1}{n}\mathcal{M}_{U_1})$ does not contain a smooth point of the exceptional divisor of β_1. Therefore, the set $\mathbb{CS}(U_1,\frac{1}{n}\mathcal{M}_{U_1})$ contains a singular point of the threefold U_1, which implies that

$$\Big\{P_1,P_i\Big\}\subseteq\mathbb{CS}\Big(Y',\frac{1}{n}\mathcal{M}_{Y'}\Big)\subseteq\Big\{P_1,\cdots,P_{15}\Big\},$$

for some $i\neq 1$. Thus, each member in the pencil $\mathcal{M}_{U_i}$ is contracted to a curve by the elliptic fibration η_i, which implies that $\mathcal{M}$ is a pencil in $|-K_X|$. □

5.3. Case $\daleth=3$, hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$.

Let X be a general hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$ with $-K_X^3=2$. It is smooth. It cannot be birationally transformed into an elliptic fibration ([4]).

Proposition 5.3.1. *Every Halphen pencil is contained in $|-K_X|$.*

Proof. It follows from Lemma 0.3.3 that the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ does not contain any point of X. Hence, it must contain a curve Z. Then, the inequality

$$\mathrm{mult}_Z(\mathcal{M})\geq n$$

holds.

For general surfaces M_1 and M_2 in $\mathcal{M}$ and a general surface D in $|-K_X|$, we have

$$2n^2=M_1\cdot M_2\cdot D\geq\mathrm{mult}^2_Z(\mathcal{M})(-K_X\cdot Z)\geq n^2,$$

which implies that there are different surfaces D_1 and D_2 in the linear system $|-K_X|$ such that the intersection $D_1\cap D_2$ contains the curve Z.

Let $\mathcal{P}$ be the pencil in $|-K_X|$ consisting of surfaces passing through the curve Z.

Suppose that $-K_X\cdot Z=2$. For a general surface D' in the pencil $\mathcal{P}$, the inequality

$$2n=M_1\cdot D'\cdot D\geq 2n$$

implies that $\mathrm{Supp}(M_1)\cap\mathrm{Supp}(D')\subset\mathrm{Supp}(Z)$. It follows from Theorem 0.2.9 that the linear system $\mathcal{M}$ is the pencil in $|-K_X|$ consisting of surfaces that pass through Z.

Now, we suppose that $-K_X\cdot Z=1$. The generality of X implies that the general surface D in $|-K_X|$ is smooth and that the intersection $D_1\cap D_2$ consists of the curve Z and an irreducible curve Z' such that $Z\neq Z'$. Hence, we have $Z^2=-2$ on the surface D and $\mathcal{M}|_D\equiv nZ+nZ'$. Therefore, the inequality $\mathrm{mult}_Z(\mathcal{M})\geq n$ implies the identity $\mathcal{M}=\mathcal{P}$ by Theorem 0.2.9. □

5.4. Case $\gimel = 4$, hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,2)$.

The weighted hypersurface X is defined by a general quasihomogeneous polynomial of degree 6 in $\mathbb{P}(1,1,1,2,2)$ with $-K_X^3 = \frac{3}{2}$. The singularities of the hypersurface X consist of points P_1, P_2, P_3 that are quotient singularities of types $\frac{1}{2}(1,1,1)$. The hypersurface X can be given by the equation

$$w^2t + \Big(t^2 + tf_2(x,y,z) + f_4(x,y,z)\Big)w + f_6(x,y,z,t) = 0$$

such that P_1 is given by the equations $x = y = z = t = 0$, where f_i is a general quasihomogeneous polynomial of degree i.

There is a commutative diagram

$$\begin{tikzcd}
 & U \arrow[dl, "\beta"'] \arrow[r, "\alpha"] & X \arrow[ddl, dashed, "\chi"'] \arrow[ddr, dashed, "\psi"] & Y \arrow[l, "\pi"'] \arrow[dd, "\eta"] \\
W \arrow[dr, "\omega"'] & & & \\
 & \mathbb{P}(1,1,1,2) \arrow[rr, dashed, "\xi"'] & & \mathbb{P}^2,
\end{tikzcd}$$

where

- ψ is the natural projection,
- π is the composition of the Kawamata blow ups at the points P_1, P_2, and P_3,
- η is an elliptic fibration
- α is the Kawamata blow up of the point P_1,
- ξ and χ are the natural projections,
- β is a birational morphism,
- ω is a double cover ramified along an octic surface $R \subset \mathbb{P}(1,1,1,2)$.

The surface R is given by the equation

$$\Big(t^2 + tf_2(x,y,z) + f_4(x,y,z)\Big)^2 - 4tf_6(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,2) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big),$$

which implies that the surface R has exactly 24 isolated ordinary double points given by the equations

$$t = t^2 + tf_2(x,y,z) + f_4(x,y,z) = f_6(x,y,z,t) = 0.$$

The birational morphism β contracts 24 smooth rational curves $C_1, \cdots, C_{24}$ to isolated ordinary double points of the variety W that dominate the singular points of R.

It easily follows from Theorems 0.2.4, 0.3.6, Lemmas 0.2.3, 0.2.7, and 0.3.3 that either the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains an irreducible curve passing through a singular point of X or the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ consists of a single singular point of X. In particular, we may assume that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains the point P_1.

Proposition 5.4.1. *Every Halphen pencil on X is contained in $|-K_X|$.*

Proof. Suppose that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains an irreducible curve Z that passes through P_1. Then, it follows from Theorem 0.3.6 that the linear system $\mathcal{M}$ is a pencil in $|-K_X|$ in the case when $-K_X \cdot Z = \frac{3}{2}$. Therefore, we may assume that the curve Z is contracted by the rational map ψ to a point. Also, we may assume that either $-K_X \cdot Z = \frac{1}{2}$ or $-K_X \cdot Z = 1$.

Let $\mathcal{B}$ be the pencil in $|-K_X|$ consisting of surfaces passing through Z. In addition, let B and B' be general surfaces in $\mathcal{B}$. Then, the cycle $B \cdot B'$ is reduced and contains the curve Z. Put $\tilde{Z} = B \cdot B'$ and let $\tilde{Z}_W$ be the image of the curve $\tilde{Z}_U$ by the birational morphism β. Then, $\omega(\tilde{Z}_W)$ is a ruling of the cone $\mathbb{P}(1,1,1,2)$. In particular, the curve $\omega(\tilde{Z}_W)$ contains at most one singular point of the surface R.

There are exactly 24 rulings of the cone $\mathbb{P}(1,1,1,2)$ that pass through the singular points of the surface R. Thus, we may assume that the curve $\tilde{Z}_W$ is irreducible in the case when the curve

$\omega(\tilde{Z}_W)$ passes through a singular point of the surface R. Moreover, the surface B_W that is the image of the surface B_U by β has an isolated ordinary double point at the point $\beta(C_i)$ in the case when $\omega\circ\beta(C_i)\in\omega(\tilde{Z}_W)$. Therefore, the cycle $\tilde{Z}$ consists of two irreducible components.

Let $\bar{Z}$ be the irreducible component of $\tilde{Z}$ that is different from Z. Then, the generality of the hypersurface X implies that $\bar{Z}^2<0$ on the surface B, but $M|_B\equiv nZ+n\bar{Z}$. On the other hand, we have

$$M\Big|_B=m_1Z+m_2\bar{Z}+F,$$

where and m_1 and m_2 are natural numbers and F is an effective divisor on B whose support contains neither the curve Z nor the curve $\bar{Z}$. We have

$$m_1\geq \mathrm{mult}_Z(\mathcal{M})\geq n$$

and

$$(n-m_2)\bar{Z}\equiv F+(m_1-n)Z,$$

They imply that $m_2=m_1=n$ and the support of the cycle $M\cdot B$ is contained in $Z\cup\bar{Z}$. Therefore, the identity $\mathcal{M}=\mathcal{B}$ follows from Theorem 0.2.9.

For now, we suppose that the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ consists of the point P_1. It follows from Lemma 0.2.6 that $\mathcal{M}_U\sim_{\mathbb{Q}}-nK_U$. Therefore, the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ is not empty by Theorem 0.2.4. Let E be the exceptional divisor of α. Then, $E\cong\mathbb{P}^2$ and the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains a line L on the surface E by Lemma 0.2.7.

Let Z be the curve $S^t_U\cap E$. Then, Z does not contain the curve L, the surface S^t_U contains every curve C_i, and the curve Z is a smooth plane quartic curve. The hypersurface X is general by assumption. In particular, the surface S^t_U is smooth along the curve C_i, the morphism $\beta|_{S^t_U}$ contracts the curve C_i to a smooth point of the surface S^t_W which is the image of S^t_U by β. Moreover, we may assume that the intersection $L\cap Z$ contains at least one point of the curve Z that is not contained in $\cup_{i=1}^{24}C_i$. Indeed, it is enough to assume that the set $\cup_{i=1}^{24}(C_i\cap Z)$ does not contain bi-tangent points of the plane quartic curve Z.

Let M' be a general surface in $\mathcal{M}$ and D be a general surface in $|-2K_U|$. Then,

$$2n^2=D\cdot M_U\cdot M'_U\geq 2\mathrm{mult}_L(M_U\cdot M'_U)\geq 2\mathrm{mult}_L(M_U)\mathrm{mult}_L(M'_U)\geq 2n^2,$$

which implies that the support of the cycle $M_U\cdot M'_U$ is contained in the union of the curve L and $\cup_{i=1}^{24}C_i$. Hence, we have

$$\mathcal{M}_U\Big|_{S^t_U}=\mathcal{D}+\sum_{i=1}^{24}m_iC_i,$$

where m_i is a natural number and $\mathcal{D}$ is a pencil without fixed components. Let P be a point of $L\cap Z$ that is not contained in $\cup_{i=1}^{24}C_i$. For general curves D_1 and D_2 in $\mathcal{D}$,

$$n^2-\sum_{i=1}^{24}m_i^2=D_1\cdot D_2\geq\mathrm{mult}_P(D_1)\mathrm{mult}_P(D_2)\geq n^2,$$

which implies that $m_1=m_2=\cdots=m_{24}=0$. Therefore, we have $M_U\cdot M'_U=n^2L$, which is impossible because the suppose of the cycle $M\cdot M'$ must contain a curve on X. $\square$

5.5. Case $\daleth=5$, hypersurface of degree 7 in $\mathbb{P}(1,1,1,2,3)$.

The threefold X is a general hypersurface of degree 7 in $\mathbb{P}(1,1,1,2,3)$ with $-K_X^3=\frac{7}{6}$. The singularities of the hypersurface X consist of two points P and Q that the are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. The hypersurface X can be given by the equation

$$w^2z+f_4(x,y,z,t)w+f_7(x,y,z,t)=0,$$

where f_i is a quasihomogeneous polynomial of degree i. Hence, the point P is given by the equations $x=y=z=w=0$ and the point Q is given by the equations $x=y=z=t=0$.

There is a commutative diagram

$$\begin{tikzcd}
& & W \arrow[dll, "\pi"'] \arrow[d, "\alpha"] & & & \\
Y \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & U \arrow[ull, "\beta"'] & \\
& \mathbb{P}(1,1,1,2) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, & & V \arrow[ll, "\eta"] \arrow[ul, "\gamma"']
\end{tikzcd}$$

where

- ψ is the natural projection,
- α is the Kawamata blow up at the point Q with weights $(1,1,2)$,
- β is the Kawamata blow up with weights $(1,1,1)$ at the point of W whose image to X is the point P,
- γ is the Kawamata blow up with weights $(1,1,1)$ at the singular point of U whose image to X is the point Q,
- η is an elliptic fibration,
- π is a birational morphism,
- χ and ξ are the natural projections,
- ω is a double cover of $\mathbb{P}(1,1,1,2)$ ramified along an octic surface R.

The generality of X implies that the birational morphism π contracts 14 smooth irreducible rational curves $C_1,\cdots,C_{14}$ into 14 isolated ordinary double points $P_1,\cdots,P_{14}$ of the variety Y, respectively. The double cover ω is branched over the octic surface $R\subset\mathbb{P}(1,1,1,2)$ that is given by the equation

$$f_4(x,y,z,t)^2-4zf_7(x,y,z,t)=0\subset\mathbb{P}(1,1,1,2)\cong\mathrm{Proj}\big(\mathbb{C}[x,y,z,t]\big),$$

which has 14 isolated ordinary double points $\omega(P_1),\cdots,\omega(P_{14})$.

Now let us prove the following result, which is due to [18].

Proposition 5.5.1. *Every Halphen pencil is contained in* $|-K_X|$.

Proof. It follows from Theorem 0.3.6 and the generality of X that the linear system $\mathcal{M}$ is a pencil in $|-K_X|$ if the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contain a curve (the proof of Proposition 5.4.1). Hence, we may assume that the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ consists of singular points of X by Lemma 0.3.3.

It easily follows from Theorem 0.2.4, Lemmas 0.2.3, and 0.2.7 that $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})\neq\{P\}$.

Suppose that $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})=\{P,Q\}$. Let E be the exceptional divisor of α. Then, the surface E is a quadric cone. It follows from Theorem 0.2.4, Lemmas 0.2.3, 0.2.6, and 0.2.7 that $\mathcal{M}_U\sim_{\mathbb{Q}}-nK_U$ and the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains a ruling L of the cone E_U. Hence, it follows from Theorem 0.2.9 and the proof of Lemma 0.2.3 imply that $\mathcal{M}_U$ is the pencil consisting of surfaces in $|-K_U|$ that contain the curve L because $-K_U\cdot L=\frac{1}{2}$ and $-K_U^3=\frac{1}{2}$.

Now, we suppose that $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})=\{Q\}$. Let O be the singular point of the variety W whose image to X is the point Q and $\bar{P}$ be the singular point of W whose image to X is the point P. It follows from Theorem 0.2.4, Lemmas 0.2.3, 0.2.6, and 0.2.7 that $\mathcal{M}_Y\sim_{\mathbb{Q}}-nK_Y$ and that the set $\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ contains an irreducible curve Z such that Z is the image of a ruling of the cone E. Hence, the equality $-K_Y\cdot Z=\frac{1}{2}$ holds and $\chi\circ\omega(Z)$ is a point.

The variety $\mathbb{P}(1,1,1,2)$ is a cone over the Veronese surface. Hence, the curve $\omega(Z)$ is a ruling of the cone $\mathbb{P}(1,1,1,2)$ and the point $(\omega\circ\pi)(\bar{P})=(\omega\circ\pi)(O)$ is the vertex of $\mathbb{P}(1,1,1,2)$. The generality of the hypersurface X implies the existence of the irreducible curve $\bar{Z}$ on the variety Y such that $\bar{Z}\neq Z$, $\omega(\bar{Z})=\omega(Z)$, and $\pi(\bar{P})\in\bar{Z}$.

Let D be a general surface in $|-K_Y|$ that contains Z. Then, the inequality $\bar{Z}^2<0$ holds on D, but $\mathcal{M}_Y|_D\equiv n\bar{Z}+nZ$. Now the proof of Proposition 5.4.1 implies that $\mathcal{M}_Y$ is the pencil consisting of surfaces in $|-K_Y|$ that contain the curve Z. $\square$

5.6. Case $\daleth = 6$, hypersurface of degree 8 in $\mathbb{P}(1,1,1,2,4)$.

The threefold X is a general hypersurface of degree 8 in $\mathbb{P}(1,1,1,2,4)$ with $-K_X^3 = 1$. Its singularities consist of points P and Q that are quotient singularities of type $\frac{1}{2}(1,1,1)$.

We have a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the composition of the Kawamata blow ups at the singular points P and Q,
- η is an elliptic fibration.

Proposition 5.6.1. *Every Halphen pencil is contained in $|-K_X|$.*

Proof. The log pair $(X, \frac{1}{n}\mathcal{M})$ has terminal singularities at a smooth point of the hypersurface X by Lemma 0.3.3. Moreover, it easily follows from Theorem 0.2.4, Lemmas 0.2.3, and 0.2.7 that the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{M})$ contains an irreducible curve Z. Then, it follows from Theorem 0.3.6 that the curve Z is a fiber of the projection ψ, which easily implies that the linear system $\mathcal{M}$ is a pencil in the linear system $|-K_X|$ by Theorem 0.2.9 in the case when the equality $-K_X \cdot Z = 1$ holds.

To conclude the proof, we assume that $-K_X \cdot Z = \frac{1}{2}$. Let D be a general surface in $|-K_X|$ that contains Z. Then, the surface D is smooth in the outside of the points P and Q which are isolated ordinary double points on D. Let F be a fiber of the rational map ψ over the point $\psi(Z)$. Then, F consists of two irreducible components. Let $\bar{Z}$ be the component of F different from Z. Then, the generality of X implies that $\bar{Z}^2 < 0$ and the proof of Proposition 5.4.1 implies that the pencil $\mathcal{M}$ consists of surfaces in $|-K_X|$ that contain the curve Z. $\square$

5.7. Case $\daleth = 8$, hypersurface of degree 9 in $\mathbb{P}(1,1,1,3,4)$.

The threefold X is a general hypersurface of degree 9 in $\mathbb{P}(1,1,1,3,4)$ with $-K_X^3 = \frac{3}{4}$. The singularities of the hypersurface X consist of the singular point O that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. The hypersurface X can be given by the equation

$$w^2z + \Big(f_2(x,y,z)t + f_5(x,y,z)\Big)w + f_9(x,y,z,t) = 0,$$

where f_i is a quasihomogeneous polynomial of degree i. Thus, the point O is given by the equations $x = y = z = t = 0$. Furthermore, we may assume that the polynomials $f_2(x,y,0)$ and $f_5(x,y,0)$ are co-prime.

There is a commutative diagram

$$\begin{tikzcd} & & W \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & U \arrow[l, "\beta"'] & \\ Y \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & V \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \\ & \mathbb{P}(1,1,1,3) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2, & \end{tikzcd}$$

where

- ξ, ψ and χ are the natural projections,
- α is the Kawamata blow up at the point O with weights $(1,1,3)$,

- β is the Kawamata blow up at the singular point of the variety W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$,
- γ is the Kawamata blow up at the singular point of the variety U that is a quotient singularity of type $\frac{1}{2}(1,1,1)$,
- η is an elliptic fibration,
- σ is a birational morphism that contracts 15 smooth rational curves to 15 isolated ordinary double points $P_1,\cdots,P_{15}$ of the variety Y,
- ω is a double cover of $\mathbb{P}(1,1,1,3)$ branched over a surface R of degree 10.

The surface R is given by the equation

$$\Big(f_2\big(x,y,z\big)t+f_5\big(x,y,z\big)\Big)^2-4zf_9\big(x,y,z,t\big)=0\subset\mathbb{P}\big(1,1,1,3\big)\cong\mathrm{Proj}\Big(\mathbb{C}[x,y,z,t]\Big).$$

It has 15 ordinary double points given by $z=tf_2+f_5=f_9=0$. Let $P_1,\cdots,P_{15}$ be the points of Y whose image via the double cover ω are the 15 ordinary double points of the surface R.

Let E be the exceptional divisor of α and F be the exceptional divisor of β. In addition, let P be the singular point of W and Q be the singular point of U. Then, the surface $\omega\circ\sigma(E)$ is given by the equation $z=0$ and $\mathbb{P}(1,1,1,3)$ is a cone whose vertex is the point $\omega\circ\sigma(P)$. The generality of the polynomials f_5 and f_2 implies that the surface R does not contain the rulings of $\mathbb{P}(1,1,1,3)$ that are contained in the surface $\omega\circ\sigma(E)$.

It follows from Lemma 0.3.3 that the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ does not contain smooth points of the hypersurface X. Therefore, by Theorem 0.3.6 and Lemma 0.2.6, it must contain the point O.

Lemma 5.7.1. *If the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains a curve, then $\mathcal{M}$ is a pencil in $|-K_X|$.*

Proof. Let L be a curve in $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$. Then, it follows from Theorem 0.3.6 that there are two different surfaces D and D' in $|-K_X|$ such that L is a component of the cycle $D\cdot D'$. Moreover, the generality of X implies that the cycle $D\cdot D'$ is reduced and contains at most two components.

Let $\mathcal{P}$ be the pencil in $|-K_X|$ generated by surfaces D and D'. From Theorem 0.2.9 and the proof of Lemma 0.2.3, we obtain $\mathcal{M}=\mathcal{P}$ if $-K_X\cdot L=\frac{3}{4}$. Hence, we may assume that either $-K_X\cdot L=\frac{1}{4}$ or $-K_X\cdot L=\frac{1}{2}$. Thus, the cycle $D\cdot D'$ contains a component L' such that

$$-K_X\cdot L'=\frac{3}{4}+K_X\cdot L$$

and $L'\neq L$. We consider only the case when $-K_X\cdot L=\frac{1}{4}$ because the case $-K_X\cdot L=\frac{1}{2}$ is simpler and similar.

The proper transform S_W^z contains the curve L_W because

$$S_W^z\sim_{\mathbb{Q}}\alpha^*(-K_X)-\frac{5}{4}E,\quad E\cdot L_W\geq\frac{1}{3}.$$

Thus, either the curve L_W is contracted by σ or the curve $\omega(L_Y)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ contained in the surface $\omega\circ\sigma(E)$, where L_Y is the image of L_W by σ.

Suppose that the curve L_W is not contracted by σ. Then, the curve $\omega(L_Y)$ is not contained in the surface R, which implies that $\omega(L_Y)$ contains at most one singular point of the surface R different from the point $\omega\circ\sigma(P)$. Moreover, the curve $\omega(L_Y)$ must contain a singular point of R different from $\omega\circ\sigma(P)$ because $-K_X\cdot L=\frac{3}{4}$ otherwise. Thus, we may assume that the curve $\omega(L_Y)$ contains the point $\omega(P_1)$.

Let $D_Y=\sigma(D_W)$ and $D_Y'=\sigma(D_W')$. Then, the point P_1 is an isolated ordinary double point of the surface D_Y. Thus, wee see that the curve L_W' is contracted to the point P_1 by σ and

$$D_Y\cdot D_Y'=L_Y+\bar{L}_Y,$$

where $\bar{L}_Y$ is a ruling of $E\cong\mathbb{P}(1,1,3)$. In particular, we have $-K_X\cdot L'=\frac{1}{4}$, which contradicts the equality $-K_X\cdot L=\frac{1}{4}$. Hence, the curve L_W is contracted by σ, which implies that the curve L_W' is not contracted by σ and the curve $\omega(L_Y')$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ that is contained in the surface $\omega\circ\sigma(E)$, where L_Y' is the image of the curve L_W' by σ. The curve $\omega(L_Y')$

is not contained in the surface R. It implies that $\omega(L'_Y)$ contains at most one singular point of the surface R different from the point $\omega\circ\sigma(P)$. The curve $\omega(L'_Y)$ must contain a singular point of R different from $\omega\circ\sigma(P)$ because $-K_X\cdot L'\neq\frac{3}{4}$. Thus, we may assume that the curve $\omega(L'_Y)$ contains the point $\omega(P_1)$.

The point P_1 is an isolated ordinary double point of the surface D_Y and the curve L_W is contracted to the point P_1 by σ. Hence, we have

$$D_Y\cdot D'_Y=L'_Y+\bar{L}'_Y,$$

where $\bar{L}'_Y$ is a ruling of $E\cong\mathbb{P}(1,1,3)$. Therefore, the intersection $L_W\cap L'_W$ consists of a point O' such that $O'\notin E$, and hence the intersection $L\cap L'$ contains the point $\alpha(O')$ that is different from O.

The surface D is normal and it is smooth at the point $\alpha(O')$. On the other hand, the equality $(L+L')\cdot L'=\frac{1}{2}$ holds on the surface D, which implies that the inequality $L'\cdot L'<0$ holds on the surface D. Therefore, we have

$$\mathcal{M}\Big|_D=m_1L+m_2L'+\mathcal{L}\equiv nL+nL',$$

where $\mathcal{L}$ is a pencil on D that does not have fixed components, and m_1 and m_2 are natural numbers such that $m_1\geq n$. In particular, we have

$$0\leq(m_1-n)L'\cdot L+\mathcal{L}\cdot L'=(n-m_2)L'\cdot L',$$

which implies that $m_2=m_1=n$ and $\mathcal{M}|_D=nL+nL'$ because $L'\cdot L'<0$. It follows from Theorem 0.2.9 that $\mathcal{M}=\mathcal{P}$. □

Therefore, we may assume that the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ consists of the singular point O, which implies that the set $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)$ contains the point P by Theorem 0.2.4 and Lemma 0.2.7.

Lemma 5.7.2. *If the set $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)$ consists of the point P, then $\mathcal{M}$ is a pencil in $|-K_X|$.*

Proof. Our assumption implies that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains the point Q by Theorem 0.2.4 and Lemma 0.2.7, and hence the set $\mathbb{CS}(V,\frac{1}{n}\mathcal{M}_V)$ is not empty by Theorem 0.2.4. However, the set $\mathbb{CS}(V,\frac{1}{n}\mathcal{M}_V)$ does not contain any subvariety of the exceptional divisor of γ by Lemmas 0.2.3 and 0.2.7. Thus, the set $\mathbb{CS}(U\frac{1}{n}\mathcal{M}_U)$ contains an element different from the point Q.

The surface F is a quadric cone and it follows from Lemma 0.2.7 that the set $\mathbb{CS}(U,\frac{1}{n}\mathcal{M}_U)$ contains a ruling L of the cone F. Let $\mathcal{B}$ be the linear system consisting of surfaces in $|-K_U|$ that contain the curve L. Then, $\mathcal{B}$ is a pencil because the curve L is contracted by the map $\eta\circ\gamma^{-1}$ to a point.

Let D be a general surface in $|-rK_U|$ for $r\gg 0$ and M_U and B be general surfaces in $\mathcal{M}_W$ and $\mathcal{B}$, respectively. Then,

$$\frac{rn}{2}=D\cdot M_U\cdot B\geq\mathrm{mult}_L(M_U\cdot B)(D\cdot L)\geq\mathrm{mult}_L(M_U)\mathrm{mult}_L(B)(D\cdot L)\geq\frac{rn}{2},$$

which implies that the support of the effective cycle $M_U\cdot B$ consists of the curve L and a cycle Z such that $D\cdot Z=0$. On the other hand, the divisor $-K_U$ is big and big. Hence, it follows from Theorem 0.2.9 that $\mathcal{M}_U=\mathcal{B}$, which implies that $\mathcal{M}$ is a pencil in $|-K_X|$. □

Proposition 5.7.3. *Every Halphen pencil is contained in $|-K_X|$*

Proof. Due to the previous arguments, we may assume that the set $\mathbb{CS}(W,\frac{1}{n}\mathcal{M}_W)$ contains a subvariety Z different from the point P. Then, Z is contained in the surface E that is a cone over the smooth rational curve of degree 3. Moreover, it follows from Lemma 0.2.7 that Z is a ruling of E. Put $\bar{Z}=\sigma(Z)$. Then, $\bar{Z}\in\mathbb{CS}(Y,\frac{1}{n}\mathcal{M}_Y)$ and $-K_Y\cdot\bar{Z}=\frac{1}{3}$.

The curve $\omega(\bar{Z})$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$. Let $\mathcal{B}$ be the linear system consisting of surfaces in $|-K_Y|$ that contain the curve $\bar{Z}$. Then, $\mathcal{B}$ is a pencil whose base locus consists of the curve $\bar{Z}$ and an irreducible smooth rational curve L on the variety Y such that L is different from the curve $\bar{Z}$ and $\omega(L)=\omega(\bar{Z})$.

Let B be a general surface in $\mathcal{B}$. Then, B is smooth in the outside of the points

$$\sigma(P)\cup\Big(\{P_1,\cdots,P_{15}\}\cap(\bar{Z}\cup L)\Big),$$

and every singular point of B different from $\sigma(P)$ is an isolated ordinary double point.

The generality of X implies that the curve $\omega(L)$ does not contain more than one singular point of R different from the point $\omega\circ\sigma(P)$. Thus, arguing as in the proof of Lemma 5.7.1, we see that the inequality $L^2<0$ holds on the surface B if the intersection $L\cap\bar{Z}$ contains a point different from $\sigma(P)$. On the other hand, the curve $\omega(L)$ does not contain singular points of R different from the point $\omega\circ\sigma(P)$ if $L\cap\bar{Z}=\{\sigma(P)\}$. Thus, the inequality $L^2<0$ holds on the surface B as well if the intersection $L\cap\bar{Z}$ consists of the point $\sigma(P)$ because the curve L is an image of the curve $\bar{Z}$ via the biregular involution of the surface B and the curve $\bar{Z}$ is contracted on the surface B.

The equivalence $\mathcal{M}_Y|_B\equiv n\bar{Z}+nL$ holds on the surface B, which implies that the support of the cycle $M_Y\cdot B$ is the union of the curves $\bar{Z}$ and L because $\mathrm{mult}_{\bar{Z}}(\mathcal{M}_Y)\geq n$. Hence, it follows from Theorem 0.2.9 that $\mathcal{M}_Y=\mathcal{B}$. Thus, the linear system $\mathcal{M}$ is a pencil in $|-K_X|$. We have completed the proof. □

5.8. Case $\gimel=10$, hypersurface of degree 10 in $\mathbb{P}(1,1,1,3,5)$.

The threefold X is a general hypersurface of degree 10 in $\mathbb{P}(1,1,1,3,5)$ with $-K_X^3=\frac{2}{3}$. The singularities of the hypersurface X consist of one point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$.

The hypersurface X can be given by the equation

$$w^2=t^3z+t^2f_4(x,y,z)+tf_7(x,y,z)+f_{10}(x,y,z),$$

where f_i is a general quasihomogeneous polynomial of degree i. In particular, we may assume that the polynomials $f_4(x,y,0)$ and $f_7(x,y,0)$ are co-prime and the polynomial $f_7^2(x,y,0)-f_4(x,y,0)f_{10}(x,y,0)$ is reduced, i.e., it has 14 distinct linear factors.

There is a commutative diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "\alpha"'] & & \\
 & X \arrow[dl, "\xi"'] \arrow[drr, dashed, "\psi"] & & & Y \arrow[ull, "\beta"'] \arrow[dl, "\eta"] \\
\mathbb{P}(1,1,1,3) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}^2, &
\end{tikzcd}$$

where

- ξ, ψ, and χ are the natural projections,
- α is the Kawamata blow up at the point O with weights $(1,1,2)$,
- β is the Kawamata blow up at the singular point of the variety W,
- η is an elliptic fibration.

Lemma 5.8.1. *If the set $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$ contains a curve, then $\mathcal{M}$ is a pencil in $|-K_X|$.*

Proof. Let L be a curve on X that is contained in $\mathbb{CS}(X,\frac{1}{n}\mathcal{M})$. Then, $-K_X\cdot L\leq\frac{2}{3}$ by Lemma 0.2.3. Moreover, the proof of Lemma 0.2.3 together with Theorem 0.2.9 and Lemma 0.3.6 implies that $\mathcal{M}$ is a pencil in $|-K_X|$ in the case when $-K_X\cdot L=\frac{2}{3}$. Hence, we may assume that $-K_X\cdot L=\frac{1}{3}$, which implies that the curve L is contracted by the rational map ψ to a point.

The variety $\mathbb{P}(1,1,1,3)$ is a cone whose vertex is the point $\xi(O)$. The curve $\xi(L)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$. The generality of the hypersurface X implies that $\xi(L)$ is not contained in the ramification divisor of ξ. Thus, there is an irreducible curve $\bar{L}$ on the variety X such that $\bar{L}$ is different from L but $\xi(L)=\xi(\bar{L})$.

Because $-K_W \cdot L_W = 0$, the curve L is one of 28 curves that are cut out on X by the equations $z = f_7^2(x,y,z) - 4f_4(x,y,z)f_{10}(x,y,z) = 0$. Therefore, the generality of the hypersurface X implies that the intersection $L \cap \bar{L}$ consists of the point O and another distinct point P.

Let $\mathcal{B}$ be the pencil consisting of surfaces in $|-K_X|$ that contain both L and $\bar{L}$ and B be a general surface in $\mathcal{B}$. Then, B is smooth at the point P. Thus, the equality $(L+\bar{L})\cdot\bar{L} = \frac{1}{3}$ holds on the surface B, which implies that $\bar{L}^2 < 0$. On the other hand, we have

$$\mathcal{M}\Big|_B = m_1 L + m_2 \bar{L} + \mathcal{L} \equiv nL + n\bar{L},$$

where $\mathcal{L}$ is a pencil on B without fixed components, and m_1 and m_2 are natural numbers such that $m_1 \geq n$. Now the inequalities $\bar{L}^2 < 0$ and $m_1 \geq n$ imply that $m_2 = m_1 = n$ and $\mathcal{M}|_B = nL + n\bar{L}$. Therefore, it follows from Theorem 0.2.9 that $\mathcal{M} = \mathcal{B}$. □

Proposition 5.8.2. *Every Halphen pencil is contained in* $|-K_X|$

Proof. Due to Lemmas 0.3.3 and 5.8.1, we may assume that $\mathbb{CS}(X, \frac{1}{n}\mathcal{M}) = \{O\}$. Let Q be the unique singular point of the variety W. Then, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains the point Q by Theorem 0.2.4 and Lemma 0.2.7.

Each member in the linear system $\mathcal{M}_Y$ is contracted to a curve by the elliptic fibration η and the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ is not empty by Theorem 0.2.4. Moreover, the set $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M}_Y)$ does not contain any subvariety of the exceptional divisor of β by Lemmas 0.2.3 and 0.2.7. Thus, the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ must contain an element other than the point Q.

Let E be the exceptional divisor of α. Then, E is a quadric cone and it follows from Lemma 0.2.7 that the set $\mathbb{CS}(W, \frac{1}{n}\mathcal{M}_W)$ contains a ruling Z of the cone E. Then, the proper transform Z_Y is contracted by η to a point.

Let $\mathcal{T}$ be the pencil consisting of surfaces in $|-K_W|$ that contain the curve Z and $\gamma : W \dashrightarrow V$ be the dominant rational map induced by the linear system $|-rK_X|$ for $r \gg 0$. The pencil $\mathcal{T}$ is the proper transform of a pencil contained in $|-K_X|$, the map γ is a birational morphism, and V is a hypersurface of degree 12 in $\mathbb{P}(1,1,1,4,6)$.

Let D be a general surface in $|-rK_X|$, and M_W and T be general surfaces in $\mathcal{M}_W$ and $\mathcal{T}$, respectively. Then,

$$\frac{rn}{2} = D \cdot M_W \cdot T \geq \mathrm{mult}_Z(M_W \cdot T)(D \cdot L) \geq \mathrm{mult}_Z(M_W)\mathrm{mult}_Z(T)(D \cdot Z) \geq \frac{rn}{2},$$

which implies that the support of the effective cycle $M_W \cdot T$ is contained in the union of the curve Z and a finite union of curves contracted by the morphism γ. Now it follows from Theorem 0.2.9 that $\mathcal{M}_W = \mathcal{T}$, which completes the proof. □

5.9. Case $\gimel = 14$, hypersurface of degree 12 in $\mathbb{P}(1,1,1,4,6)$.

Let X be a general hypersurface of degree 12 in $\mathbb{P}(1,1,1,4,6)$ with $-K_X^3 = \frac{1}{2}$. It has only one singular point P that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

We have an elliptic fibration as follows:

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2, \end{tikzcd}$$

where

- ψ is the natural projection,
- π is the Kawamata blow up at the point P with weights $(1,1,1)$,
- η is an elliptic fibration.

Proposition 5.9.1. *Every Halphen pencil on X is contained in* $|-K_X|$.

Proof. The log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal by Theorem 0.2.4. However, it is terminal at a smooth point by Lemma 0.3.3.

Suppose that the log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal along a curve $Z \subset X$. Then, the inequality

$$\mathrm{mult}_Z(\mathcal{M}) \geq n$$

holds.

For general surfaces M_1 and M_2 in $\mathcal{M}$ and a general surface D in $|-K_X|$, we have

$$\frac{n^2}{2} = M_1 \cdot M_2 \cdot D \geq \mathrm{mult}_Z^2(\mathcal{M})(-K_X \cdot Z) \geq \frac{n^2}{2},$$

which implies that the curve Z is a fiber of the rational map ψ. For a general surface D' in $|-K_X|$ that contains the curve Z,

$$\frac{n}{2} = M_1 \cdot D \cdot D' \geq \frac{n}{2},$$

which implies that $\mathrm{Supp}(M_1) \cap \mathrm{Supp}(D') \subset \mathrm{Supp}(Z)$. It follows from Theorem 0.2.9 that the linear system $\mathcal{M}$ is the pencil in $|-K_X|$ consisting of surfaces that pass through Z.

Suppose that the log pair $(X, \frac{1}{n}\mathcal{M})$ is not terminal at the point P. Because $-K_Y^3 = 0$ and $\mathcal{M}_Y \sim_{\mathbb{Q}} -nK_Y$, every surface in the pencil $\mathcal{M}_Y$ is contracted to a curve by the morphism η. The log pair $(Y, \frac{1}{n}\mathcal{M}_Y)$ is not terminal along a curve contained in the exceptional divisor of π by Theorem 0.2.4 and Lemma 0.2.7. However, it contradicts Lemma 0.2.3. $\square$

Part 6. The Table.

Before we explain the table, we should mention that all the contents, except the numbers of Halphen pencils, are obtained from [7].

We tabulate the singular points of the hypersurface

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4),$$

and the number of Halphen pencils on X, *i.e.*, the number of ways in which the hypersurface X is birationally transformed into a fibration by surfaces of Kodaira dimension zero.

The contents in the entries on the first row and the second column is the number of Halphen pencils. These pencils define rational maps a generic fiber of which is birational to a smooth K3 surface.

The contents in the entries from the second rows explain the singular points on X. The first column tabulates the types of singularities. The second column shows the numbers b and c in Proposition 0.3.9 when we take the Kawamata blow up at a given point. The divisors B and E are the anticanonical divisor and the exceptional divisor , respectively, on the Kawamata blow up at a given singular point. For simplicity, we keep the divisors $bB + cE$ only when $B^3 < 0$. The blank entries simply mean $B^3 \geq 0$. In such cases, we do not need the divisors $bB + cE$ for the present article.

$\gimel = 1$: $X_4 \subset \mathbb{P}(1,1,1,1,1)$

$-K_X^3$=4	∞
smooth	N/A

$\gimel = 2$: $X_5 \subset \mathbb{P}(1,1,1,1,2)$

$-K_X^3$=5/2	∞
$P_4 = \frac{1}{2}(1,1,1)$	

$\gimel = 3$: $X_6 \subset \mathbb{P}(1,1,1,1,3)$

$-K_X^3$=2	∞
smooth	N/A

$\gimel = 4$: $X_6 \subset \mathbb{P}(1,1,1,2,2)$

$-K_X^3$=3/2	∞
$P_3P_4 = 3 \times \frac{1}{2}(1,1,1)$	

$\gimel = 5$: $X_7 \subset \mathbb{P}(1,1,1,2,3)$

$-K_X^3$=7/6	∞
$P_4 = \frac{1}{3}(1,1,2)$	
$P_3 = \frac{1}{2}(1,1,1)$	

$\gimel = 6$: $X_8 \subset \mathbb{P}(1,1,1,2,4)$

$-K_X^3$=1	∞
$P_3P_4 = 2 \times \frac{1}{2}(1,1,1)$	

$\gimel = 7$: $X_8 \subset \mathbb{P}(1,1,2,2,3)$

$-K_X^3$=2/3	1
$P_4 = \frac{1}{3}(1,1,2)$	
$P_2P_3 = 4 \times \frac{1}{2}(1,1,1)$	

$\gimel = 8$: $X_9 \subset \mathbb{P}(1,1,1,3,4)$

$-K_X^3$=3/4	∞
$P_4 = \frac{1}{4}(1,1,3)$	

$\beth = 9$: $X_9 \subset \mathbb{P}(1,1,2,3,3)$

$-K_X^3$=1/2	1
$P_2 = \frac{1}{2}(1,1,1)$	
$P_3P_4 = 3 \times \frac{1}{3}(1,1,2)$	

$\beth = 10$: $X_{10} \subset \mathbb{P}(1,1,1,3,5)$

$-K_X^3$=2/3	∞
$P_3 = \frac{1}{3}(1,1,2)$	

$\beth = 11$: $X_{10} \subset \mathbb{P}(1,1,2,2,5)$

$-K_X^3$=1/2	1
$P_2P_3 = 5 \times \frac{1}{2}(1,1,1)$	

$\beth = 12$: $X_{10} \subset \mathbb{P}(1,1,2,3,4)$

$-K_X^3$=5/12	1
$P_4 = \frac{1}{4}(1,1,3)$	
$P_3 = \frac{1}{3}(1,1,2)$	
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	B

$\beth = 13$: $X_{11} \subset \mathbb{P}(1,1,2,3,5)$

$-K_X^3$=11/30	1
$P_4 = \frac{1}{5}(1,2,3)$	
$P_3 = \frac{1}{3}(1,1,2)$	
$P_2 = \frac{1}{2}(1,1,1)$	B

$\beth = 14$: $X_{12} \subset \mathbb{P}(1,1,1,4,6)$

$-K_X^3$=1/2	∞
$P_3P_4 = 1 \times \frac{1}{2}(1,1,1)$	

$\beth = 15$: $X_{12} \subset \mathbb{P}(1,1,2,3,6)$

$-K_X^3$=1/3	1
$P_3P_4 = 2 \times \frac{1}{3}(1,1,2)$	
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	B

$\beth = 16$: $X_{12} \subset \mathbb{P}(1,1,2,4,5)$

$-K_X^3$=3/10	1
$P_4 = \frac{1}{5}(1,1,4)$	
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	B

$\beth = 17$: $X_{12} \subset \mathbb{P}(1,1,3,4,4)$

$-K_X^3$=1/4	1
$P_3P_4 = 3 \times \frac{1}{4}(1,1,3)$	

$\beth = 18$: $X_{12} \subset \mathbb{P}(1,2,2,3,5)$

$-K_X^3$=1/5	7
$P_4 = \frac{1}{5}(1,2,3)$	
$P_1P_2 = 6 \times \frac{1}{2}(1,1,1)$	$2B$

$\gimel = 19$: $X_{12} \subset \mathbb{P}(1,2,3,3,4)$

$-K_X^3$=1/6	1
$P_1P_4 = 3 \times \frac{1}{2}(1,1,1)$	$6B+E$
$P_2P_3 = 4 \times \frac{1}{3}(1,2,1)$	

$\gimel = 20$: $X_{13} \subset \mathbb{P}(1,1,3,4,5)$

$-K_X^3$=13/60	1
$P_4 = \frac{1}{5}(1,1,4)$	
$P_3 = \frac{1}{4}(1,1,3)$	
$P_2 = \frac{1}{3}(1,1,2)$	

$\gimel = 21$: $X_{14} \subset \mathbb{P}(1,1,2,4,7)$

$-K_X^3$=1/4	1
$P_3 = \frac{1}{4}(1,1,3)$	
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	B

$\gimel = 22$: $X_{14} \subset \mathbb{P}(1,2,2,3,7)$

$-K_X^3$=1/6	8
$P_3 = \frac{1}{3}(1,2,1)$	
$P_1P_2 = 7 \times \frac{1}{2}(1,1,1)$	$2B$

$\gimel = 23$: $X_{14} \subset \mathbb{P}(1,2,3,4,5)$

$-K_X^3$=7/60	1
$P_4 = \frac{1}{5}(1,2,3)$	
$P_3 = \frac{1}{4}(1,3,1)$	
$P_2 = \frac{1}{3}(1,2,1)$	$2B$
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$4B+E$

$\gimel = 24$: $X_{15} \subset \mathbb{P}(1,1,2,5,7)$

$-K_X^3$=3/14	1
$P_4 = \frac{1}{7}(1,2,5)$	
$P_2 = \frac{1}{2}(1,1,1)$	B

$\gimel = 25$: $X_{15} \subset \mathbb{P}(1,1,3,4,7)$

$-K_X^3$=5/28	1
$P_4 = \frac{1}{7}(1,3,4)$	
$P_3 = \frac{1}{4}(1,1,3)$	

$\gimel = 26$: $X_{15} \subset \mathbb{P}(1,1,3,5,6)$

$-K_X^3$=1/6	1
$P_4 = \frac{1}{6}(1,1,5)$	
$P_3P_4 = 2 \times \frac{1}{3}(1,1,2)$	

$\gimel = 27$: $X_{15} \subset \mathbb{P}(1,2,3,5,5)$

$-K_X^3$=1/10	1
$P_3P_4 = 3 \times \frac{1}{5}(1,2,3)$	
$P_1 = \frac{1}{2}(1,1,1)$	$5B+E$

$\gimel = 28$: $X_{15} \subset \mathbb{P}(1,3,3,4,5)$

$-K_X^3$=1/12	6
$P_3 = \frac{1}{4}(1,3,1)$	
$P_1P_2 = 5 \times \frac{1}{3}(1,1,2)$	$3B$

$\gimel = 29$: $X_{16} \subset \mathbb{P}(1,1,2,5,8)$

$-K_X^3$=1/5	1
$P_3 = \frac{1}{5}(1,2,3)$	
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	B

$\gimel = 30$: $X_{16} \subset \mathbb{P}(1,1,3,4,8)$

$-K_X^3$=1/6	1
$P_3P_4 = 2 \times \frac{1}{4}(1,1,3)$	
$P_2 = \frac{1}{3}(1,1,2)$	

$\gimel = 31$: $X_{16} \subset \mathbb{P}(1,1,4,5,6)$

$-K_X^3$=2/15	1
$P_4 = \frac{1}{6}(1,1,5)$	
$P_3 = \frac{1}{5}(1,1,4)$	
$P_2P_4 = 1 \times \frac{1}{2}(1,1,1)$	B

$\gimel = 32$: $X_{16} \subset \mathbb{P}(1,2,3,4,7)$

$-K_X^3$=2/21	1
$P_4 = \frac{1}{7}(1,3,4)$	
$P_2 = \frac{1}{3}(1,2,1)$	$2B$
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$4B+E$

$\gimel = 33$: $X_{17} \subset \mathbb{P}(1,2,3,5,7)$

$-K_X^3$=17/210	1
$P_4 = \frac{1}{7}(1,2,5)$	
$P_3 = \frac{1}{5}(1,2,3)$	
$P_2 = \frac{1}{3}(1,2,1)$	$2B$
$P_1 = \frac{1}{2}(1,1,1)$	$7B+2E$

$\gimel = 34$: $X_{18} \subset \mathbb{P}(1,1,2,6,9)$

$-K_X^3$=1/6	1
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	B

$\gimel = 35$: $X_{18} \subset \mathbb{P}(1,1,3,5,9)$

$-K_X^3$=2/15	1
$P_3 = \frac{1}{5}(1,1,4)$	
$P_2P_4 = 2 \times \frac{1}{3}(1,1,2)$	B

$\gimel = 36$: $X_{18} \subset \mathbb{P}(1,1,4,6,7)$

$-K_X^3$=3/28	1
$P_4 = \frac{1}{7}(1,1,6)$	
$P_2 = \frac{1}{4}(1,1,3)$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	B

$\gimel = 37$: $X_{18} \subset \mathbb{P}(1,2,3,4,9)$

$-K_X^3$=1/12	1
$P_3 = \frac{1}{4}(1,3,1)$	
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$2B$
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$4B+E$

$\gimel = 38$: $X_{18} \subset \mathbb{P}(1,2,3,5,8)$

$-K_X^3$=3/40	1
$P_4 = \frac{1}{8}(1,3,5)$	
$P_3 = \frac{1}{5}(1,2,3)$	
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$10B+3E$

$\gimel = 39$: $X_{18} \subset \mathbb{P}(1,3,4,5,6)$

$-K_X^3$=1/20	1
$P_3 = \frac{1}{5}(1,4,1)$	
$P_2 = \frac{1}{4}(1,3,1)$	$3B$
$P_2P_4 = \frac{1}{2}(1,1,1)$	$3B+E$
$P_1P_4 = 3 \times \frac{1}{3}(1,1,2)$	$20B+3E$

$\gimel = 40$: $X_{19} \subset \mathbb{P}(1,3,4,5,7)$

$-K_X^3$=19/420	1
$P_4 = \frac{1}{7}(1,3,4)$	
$P_3 = \frac{1}{5}(1,3,2)$	
$P_2 = \frac{1}{4}(1,3,1)$	$3B$
$P_1 = \frac{1}{3}(1,1,2)$	$7B+E$

$\gimel = 41$: $X_{20} \subset \mathbb{P}(1,1,4,5,10)$

$-K_X^3$=1/10	1
$P_3P_4 = 2 \times \frac{1}{5}(1,1,4)$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	B

$\gimel = 42$: $X_{20} \subset \mathbb{P}(1,2,3,5,10)$

$-K_X^3$=1/15	1
$P_2 = \frac{1}{3}(1,2,1)$	$2B$
$P_3P_4 = 2 \times \frac{1}{5}(1,2,3)$	
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$10B+3E$

$\gimel = 43$: $X_{20} \subset \mathbb{P}(1,2,4,5,9)$

$-K_X^3$=1/18	1
$P_4 = \frac{1}{9}(1,4,5)$	
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$4B+E$

$\gimel = 44$: $X_{20} \subset \mathbb{P}(1,2,5,6,7)$

$-K_X^3$=1/21	1
$P_4 = \frac{1}{7}(1,2,5)$	
$P_3 = \frac{1}{6}(1,5,1)$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$6B + 2E$

$\gimel = 45$: $X_{20} \subset \mathbb{P}(1,3,4,5,8)$

$-K_X^3$=1/24	2
$P_4 = \frac{1}{8}(1,3,5)$	
$P_1 = \frac{1}{3}(1,1,2)$	$8B + E$
$P_2P_4 = 2 \times \frac{1}{4}(1,3,1)$	$3B$

$\gimel = 46$: $X_{21} \subset \mathbb{P}(1,1,3,7,10)$

$-K_X^3$=1/10	1
$P_4 = \frac{1}{10}(1,3,7)$	

$\gimel = 47$: $X_{21} \subset \mathbb{P}(1,1,5,7,8)$

$-K_X^3$=3/40	1
$P_4 = \frac{1}{8}(1,1,7)$	
$P_2 = \frac{1}{5}(1,2,3)$	

$\gimel = 48$: $X_{21} \subset \mathbb{P}(1,2,3,7,9)$

$-K_X^3$=1/18	2
$P_4 = \frac{1}{9}(1,2,7)$	
$P_1 = \frac{1}{2}(1,1,1)$	$3B + E$
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$2B$

$\gimel = 49$: $X_{21} \subset \mathbb{P}(1,3,5,6,7)$

$-K_X^3$=1/30	1
$P_3 = \frac{1}{6}(1,5,1)$	
$P_2 = \frac{1}{5}(1,3,2)$	
$P_1P_3 = 3 \times \frac{1}{3}(1,2,1)$	$7B + E$

$\gimel = 50$: $X_{22} \subset \mathbb{P}(1,1,3,7,11)$

$-K_X^3$=2/21	1
$P_3 = \frac{1}{7}(1,3,4)$	
$P_2 = \frac{1}{3}(1,1,2)$	B

$\gimel = 51$: $X_{22} \subset \mathbb{P}(1,1,4,6,11)$

$-K_X^3$=1/12	1
$P_3 = \frac{1}{6}(1,1,5)$	
$P_2 = \frac{1}{4}(1,1,3)$	
$P_2P_3 = \frac{1}{2}(1,1,1)$	B

$\gimel = 52$: $X_{22} \subset \mathbb{P}(1,2,4,5,11)$

$-K_X^3$=1/20	1
$P_3 = \frac{1}{5}(1,4,1)$	
$P_2 = \frac{1}{4}(1,1,3)$	$2B$
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$4B+E$

$\gimel = 53$: $X_{24} \subset \mathbb{P}(1,1,3,8,12)$

$-K_X^3$=1/12	1
$P_3P_4 = 1 \times \frac{1}{4}(1,1,3)$	
$P_2P_4 = 2 \times \frac{1}{3}(1,1,2)$	B

$\gimel = 54$: $X_{24} \subset \mathbb{P}(1,1,6,8,9)$

$-K_X^3$=1/18	1
$P_4 = \frac{1}{9}(1,1,8)$	
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	B
$P_2P_3 = \frac{1}{2}(1,1,1)$	B

$\gimel = 55$: $X_{24} \subset \mathbb{P}(1,2,3,7,12)$

$-K_X^3$=1/21	2
$P_3 = \frac{1}{7}(1,2,5)$	
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$2B$
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$3B+E$

$\gimel = 56$: $X_{24} \subset \mathbb{P}(1,2,3,8,11)$

$-K_X^3$=1/22	1
$P_4 = \frac{1}{11}(1,3,8)$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$3B+E$

$\gimel = 57$: $X_{24} \subset \mathbb{P}(1,3,4,5,12)$

$-K_X^3$=1/30	2
$P_3 = \frac{1}{5}(1,3,2)$	
$P_2P_4 = 2 \times \frac{1}{4}(1,3,1)$	$3B$
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$12B+2E$

$\gimel = 58$: $X_{24} \subset \mathbb{P}(1,3,4,7,10)$

$-K_X^3$=1/35	2
$P_4 = \frac{1}{10}(1,3,7)$	
$P_3 = \frac{1}{7}(1,3,4)$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	$3B+E$

$\gimel = 59$: $X_{24} \subset \mathbb{P}(1,3,6,7,8)$

$-K_X^3$=1/42	1
$P_3 = \frac{1}{7}(1,6,1)$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	$3B+E$
$P_1P_2 = 4 \times \frac{1}{3}(1,1,2)$	$6B+E$

$\daleth = 60$: $X_{24} \subset \mathbb{P}(1,4,5,6,9)$

$-K_X^3$=1/45	2
$P_4 = \frac{1}{9}(1,4,5)$	
$P_2 = \frac{1}{5}(1,4,1)$	$4B$
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$5B+E$
$P_1P_3 = 2 \times \frac{1}{2}(1,1,1)$	$18B+7E$

$\daleth = 61$: $X_{25} \subset \mathbb{P}(1,4,5,7,9)$

$-K_X^3$=5/252	1
$P_4 = \frac{1}{9}(1,4,5)$	
$P_3 = \frac{1}{7}(1,5,2)$	
$P_1 = \frac{1}{4}(1,3,1)$	$9B+E$

$\daleth = 62$: $X_{26} \subset \mathbb{P}(1,1,5,7,13)$

$-K_X^3$=2/35	1
$P_3 = \frac{1}{7}(1,1,6)$	
$P_2 = \frac{1}{5}(1,2,3)$	

$\daleth = 63$: $X_{26} \subset \mathbb{P}(1,2,3,8,13)$

$-K_X^3$=1/24	1
$P_3 = \frac{1}{8}(1,3,5)$	
$P_2 = \frac{1}{3}(1,2,1)$	$2B$
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$3B+E$

$\daleth = 64$: $X_{26} \subset \mathbb{P}(1,2,5,6,13)$

$-K_X^3$=1/30	1
$P_3 = \frac{1}{6}(1,5,1)$	
$P_2 = \frac{1}{5}(1,2,3)$	
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$6B+2E$

$\daleth = 65$: $X_{27} \subset \mathbb{P}(1,2,5,9,11)$

$-K_X^3$=3/110	1
$P_4 = \frac{1}{11}(1,2,9)$	
$P_2 = \frac{1}{5}(1,4,1)$	$2B$
$P_1 = \frac{1}{2}(1,1,1)$	$11B+4E$

$\daleth = 66$: $X_{27} \subset \mathbb{P}(1,5,6,7,9)$

$-K_X^3$=1/70	2
$P_3 = \frac{1}{7}(1,5,2)$	
$P_2 = \frac{1}{6}(1,5,1)$	$5B$
$P_1 = \frac{1}{5}(1,1,4)$	$12B+E$
$P_2P_4 = 1 \times \frac{1}{3}(1,2,1)$	$5B+E$

$\daleth = 67$: $X_{28} \subset \mathbb{P}(1,1,4,9,14)$

$-K_X^3$=1/18	1
$P_3 = \frac{1}{9}(1,4,5)$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	B

$\daleth = 68$: $X_{28} \subset \mathbb{P}(1,3,4,7,14)$

$-K_X^3$=1/42	1
$P_1 = \frac{1}{3}(1,1,2)$	$7B+E$
$P_3P_4 = 2 \times \frac{1}{7}(1,3,4)$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	$3B+E$

$\daleth = 69$: $X_{28} \subset \mathbb{P}(1,4,6,7,11)$

$-K_X^3$=1/66	2
$P_4 = \frac{1}{11}(1,4,7)$	
$P_2 = \frac{1}{6}(1,1,5)$	$4B$
$P_1P_2 = 2 \times \frac{1}{2}(1,1,1)$	$22B+9E$

$\daleth = 70$: $X_{30} \subset \mathbb{P}(1,1,4,10,15)$

$-K_X^3$=1/20	1
$P_2 = \frac{1}{4}(1,1,3)$	B
$P_3P_4 = \frac{1}{5}(1,1,4)$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	B

$\daleth = 71$: $X_{30} \subset \mathbb{P}(1,1,6,8,15)$

$-K_X^3$=1/24	1
$P_3 = \frac{1}{8}(1,1,7)$	
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	B
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	B

$\daleth = 72$: $X_{30} \subset \mathbb{P}(1,2,3,10,15)$

$-K_X^3$=1/30	1
$P_3P_4 = 1 \times \frac{1}{5}(1,2,3)$	
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$2B$
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$3B+E$

$\daleth = 73$: $X_{30} \subset \mathbb{P}(1,2,6,7,15)$

$-K_X^3$=1/42	1
$P_3 = \frac{1}{7}(1,6,1)$	
$P_2P_4 = 1 \times \frac{1}{3}(1,2,1)$	$2B$
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$6B+2E$

$\daleth = 74$: $X_{30} \subset \mathbb{P}(1,3,4,10,13)$

$-K_X^3$=1/52	2
$P_4 = \frac{1}{13}(1,3,10)$	
$P_2 = \frac{1}{4}(1,3,1)$	$3B$
$P_2P_3 = \frac{1}{2}(1,1,1)$	$3B+E$

$\daleth = 75$: $X_{30} \subset \mathbb{P}(1,4,5,6,15)$

$-K_X^3$=1/60	1
$P_1 = \frac{1}{4}(1,1,3)$	$10B+E$
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$5B+E$
$P_2P_4 = 2 \times \frac{1}{5}(1,4,1)$	$4B$
$P_1P_3 = 2 \times \frac{1}{2}(1,1,1)$	$5B+2E$

$\daleth = 76$: $X_{30} \subset \mathbb{P}(1,5,6,8,11)$

$-K_X^3$=1/88	2
$P_4 = \frac{1}{11}(1,5,6)$	
$P_3 = \frac{1}{8}(1,5,3)$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

$\daleth = 77$: $X_{32} \subset \mathbb{P}(1,2,5,9,16)$

$-K_X^3$=1/45	1
$P_3 = \frac{1}{9}(1,2,7)$	
$P_2 = \frac{1}{5}(1,4,1)$	$2B$
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$18B + 7E$

$\daleth = 78$: $X_{32} \subset \mathbb{P}(1,4,5,7,16)$

$-K_X^3$=1/70	1
$P_3 = \frac{1}{7}(1,5,2)$	
$P_2 = \frac{1}{5}(1,4,1)$	$4B$
$P_1P_4 = 2 \times \frac{1}{4}(1,1,3)$	$7B + E$

$\daleth = 79$: $X_{33} \subset \mathbb{P}(1,3,5,11,14)$

$-K_X^3$=1/70	2
$P_4 = \frac{1}{14}(1,3,11)$	
$P_2 = \frac{1}{5}(1,1,4)$	$3B$

$\daleth = 80$: $X_{34} \subset \mathbb{P}(1,3,4,10,17)$

$-K_X^3$=1/60	2
$P_3 = \frac{1}{10}(1,3,7)$	
$P_2 = \frac{1}{4}(1,3,1)$	$3B$
$P_1 = \frac{1}{3}(1,1,2)$	$10B + 2E$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$3B + E$

$\daleth = 81$: $X_{34} \subset \mathbb{P}(1,4,6,7,17)$

$-K_X^3$=1/84	2
$P_3 = \frac{1}{7}(1,4,3)$	
$P_2 = \frac{1}{6}(1,1,5)$	$4B$
$P_1 = \frac{1}{4}(1,3,1)$	$7B + E$
$P_1P_2 = 2 \times \frac{1}{2}(1,1,1)$	$12B + 5E$

$\daleth = 82$: $X_{36} \subset \mathbb{P}(1,1,5,12,18)$

$-K_X^3$=1/30	1
$P_2 = \frac{1}{5}(1,2,3)$	
$P_3P_4 = 1 \times \frac{1}{6}(1,1,5)$	

$\daleth = 83$: $X_{36} \subset \mathbb{P}(1,3,4,11,18)$

$-K_X^3$=1/66	1
$P_3 = \frac{1}{11}(1,4,7)$	
$P_2P_4 = 1 \times \frac{1}{2}(1,1,1)$	$3B + E$
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$6B + 18E$

$\daleth = 84$: $X_{36} \subset \mathbb{P}(1,7,8,9,12)$

$-K_X^3$=1/168	2
$P_2 = \frac{1}{8}(1,7,1)$	$7B$
$P_1 = \frac{1}{7}(1,2,5)$	$8B$
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$8B + 2E$
$P_2P_4 = 1 \times \frac{1}{4}(1,3,1)$	$7B + E$

$\daleth = 85$: $X_{38} \subset \mathbb{P}(1,3,5,11,19)$

$-K_X^3$=2/165	1
$P_3 = \frac{1}{11}(1,3,8)$	
$P_2 = \frac{1}{5}(1,1,4)$	$3B$
$P_1 = \frac{1}{3}(1,2,1)$	$5B + E$

$\daleth = 86$: $X_{38} \subset \mathbb{P}(1,5,6,8,19)$

$-K_X^3$=1/120	2
$P_3 = \frac{1}{8}(1,5,3)$	
$P_2 = \frac{1}{6}(1,5,1)$	$5B$
$P_1 = \frac{1}{5}(1,1,4)$	$18B + 2E$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

$\daleth = 87$: $X_{40} \subset \mathbb{P}(1,5,7,8,20)$

$-K_X^3$=1/140	1
$P_2 = \frac{1}{7}(1,1,6)$	$5B$
$P_3P_4 = 1 \times \frac{1}{4}(1,1,3)$	$7B + E$
$P_1P_4 = 2 \times \frac{1}{5}(1,2,3)$	$20B + 3E$

$\daleth = 88$: $X_{42} \subset \mathbb{P}(1,1,6,14,21)$

$-K_X^3$=1/42	1
$P_3P_4 = 1 \times \frac{1}{7}(1,1,6)$	
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	B
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	B

$\daleth = 89$: $X_{42} \subset \mathbb{P}(1,2,5,14,21)$

$-K_X^3$=1/70	1
$P_2 = \frac{1}{5}(1,4,1)$	$2B$
$P_3P_4 = 1 \times \frac{1}{7}(1,2,5)$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

$\daleth = 90$: $X_{42} \subset \mathbb{P}(1,3,4,14,21)$

$-K_X^3$=1/84	1
$P_2 = \frac{1}{4}(1,3,1)$	$3B$
$P_3P_4 = \frac{1}{7}(1,3,4)$	
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$21B + 5E$
$P_2P_3 = \frac{1}{2}(1,1,1)$	$3B + E$

$\gimel = 91$: $X_{44} \subset \mathbb{P}(1,4,5,13,22)$

$-K_X^3$=1/130	2
$P_3 = \frac{1}{13}(1,4,9)$	
$P_2 = \frac{1}{5}(1,3,2)$	$4B$
$P_1P_4 = 1 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

$\gimel = 92$: $X_{48} \subset \mathbb{P}(1,3,5,16,24)$

$-K_X^3$=1/120	1
$P_2 = \frac{1}{5}(1,1,4)$	$3B$
$P_3P_4 = 1 \times \frac{1}{8}(1,3,5)$	
$P_1P_4 = 2 \times \frac{1}{3}(1,2,1)$	$5B + E$

$\gimel = 93$: $X_{50} \subset \mathbb{P}(1,7,8,10,25)$

$-K_X^3$=1/280	2
$P_2 = \frac{1}{8}(1,7,1)$	$7B$
$P_1 = \frac{1}{7}(1,3,4)$	$8B$
$P_3P_4 = 1 \times \frac{1}{5}(1,2,3)$	$8B + E$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$7B + 3E$

$\gimel = 94$: $X_{54} \subset \mathbb{P}(1,4,5,18,27)$

$-K_X^3$=1/180	1
$P_2 = \frac{1}{5}(1,3,2)$	$4B$
$P_1 = \frac{1}{4}(1,1,3)$	$18B + 3E$
$P_3P_4 = 1 \times \frac{1}{9}(1,4,5)$	
$P_1P_3 = 1 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

$\gimel = 95$: $X_{66} \subset \mathbb{P}(1,5,6,22,33)$

$-K_X^3$=1/330	2
$P_1 = \frac{1}{5}(1,2,3)$	$6B$
$P_3P_4 = 1 \times \frac{1}{11}(1,5,6)$	
$P_2P_4 = \frac{1}{3}(1,2,1)$	$5B + E$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$5B + 2E$

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